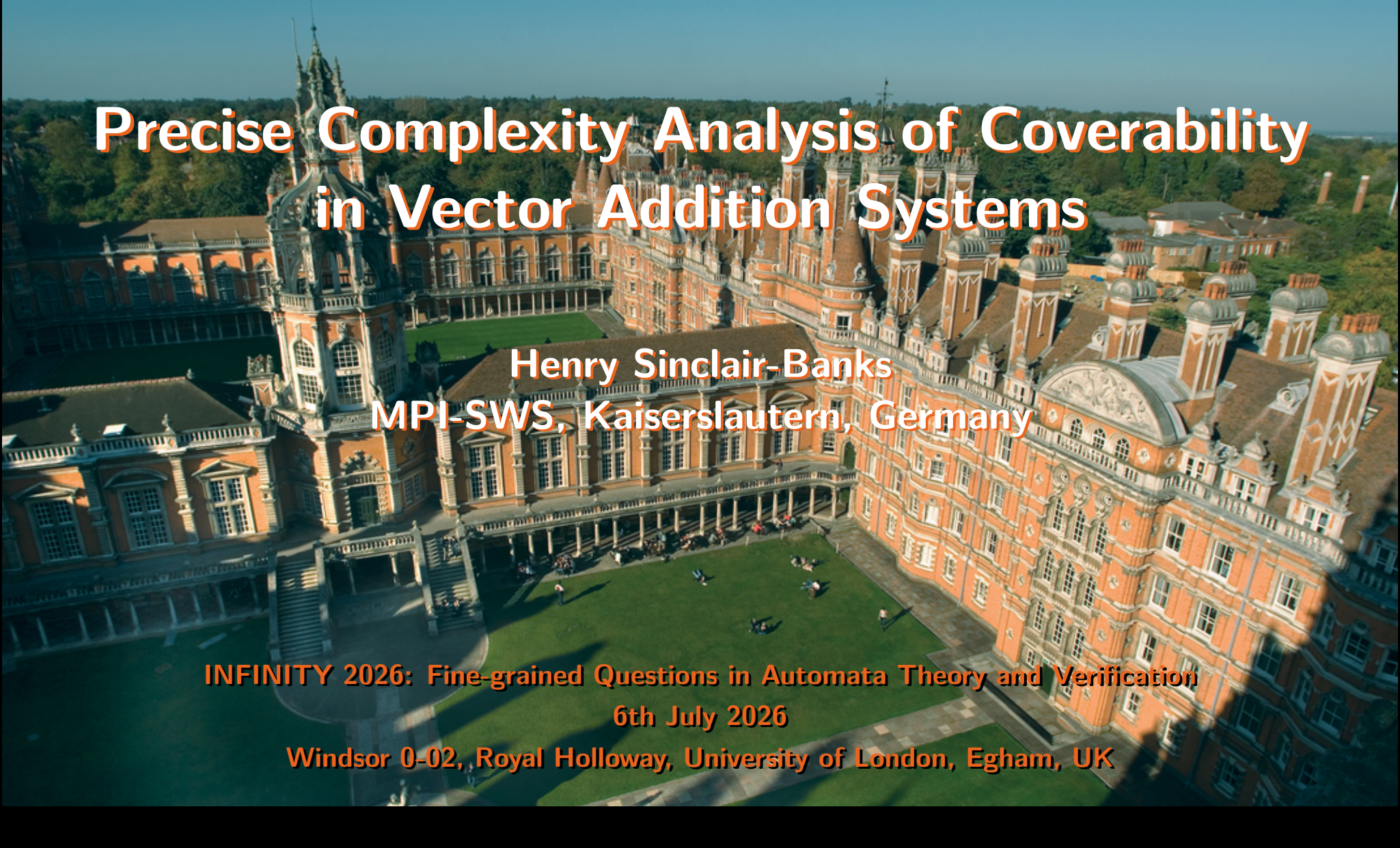

Precise Complexity Analysis of Coverability in Vector Addition Systems

Henry Sinclair-Banks
MPI-SWS, Kaiserslautern, Germany

INFINITY 2026: Fine-grained Questions in Automata Theory and Verification
6th July 2026
Windsor 0-02, Royal Holloway, University of London, Egham, UK

Vector Addition Systems with States

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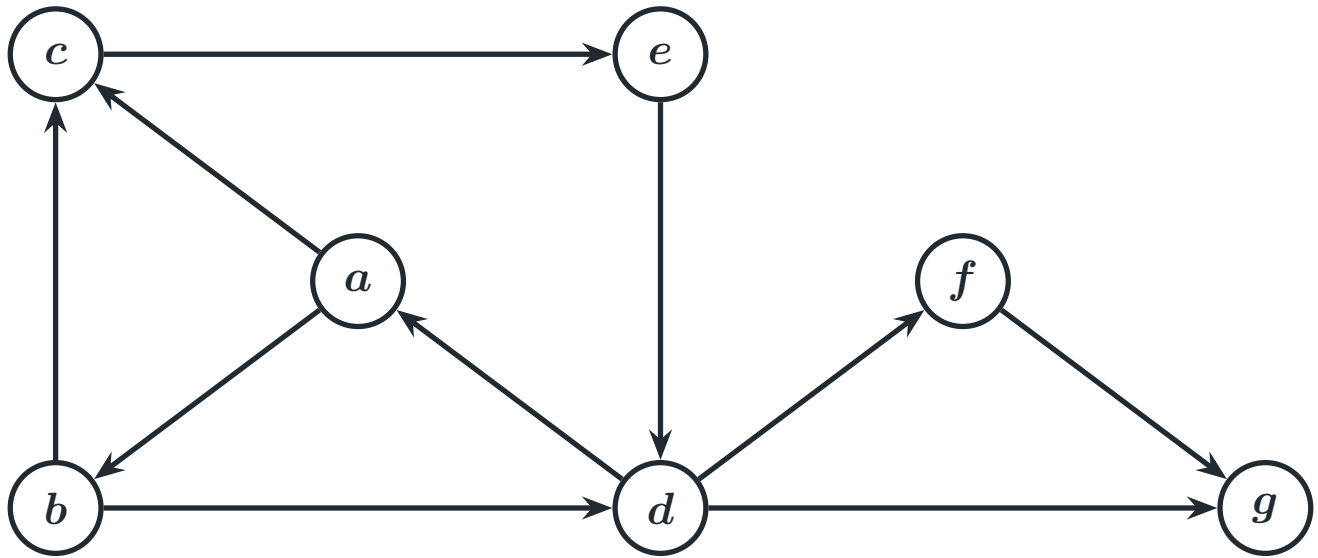

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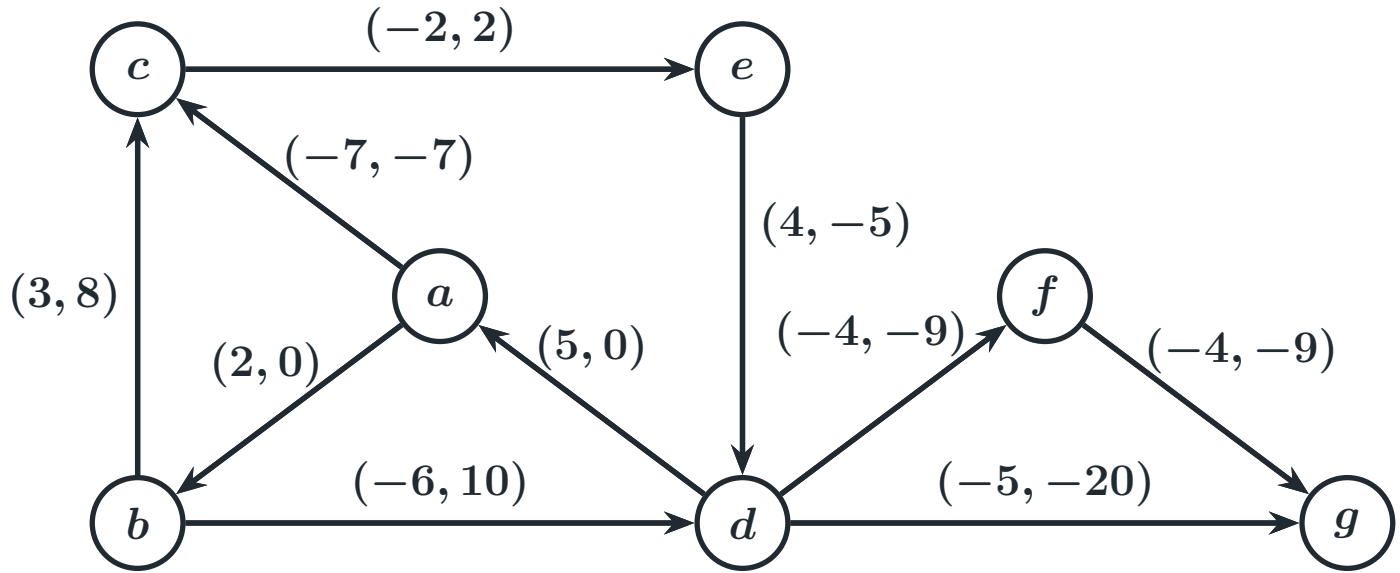
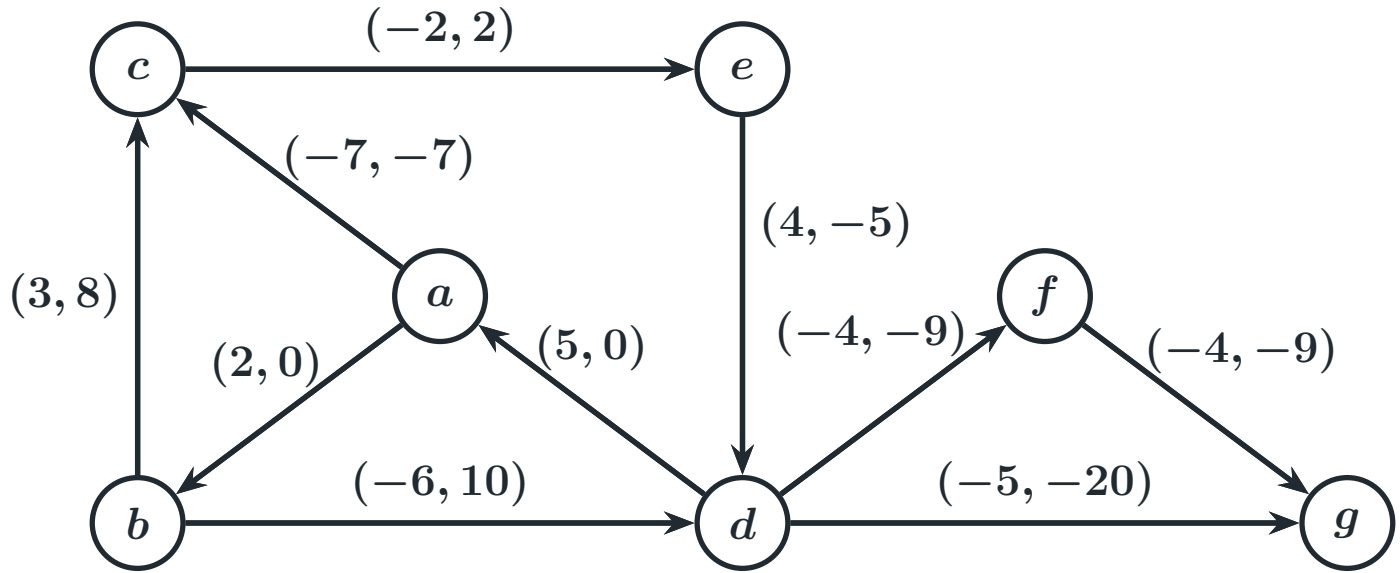


2-dimensional Vector Addition Systems with States



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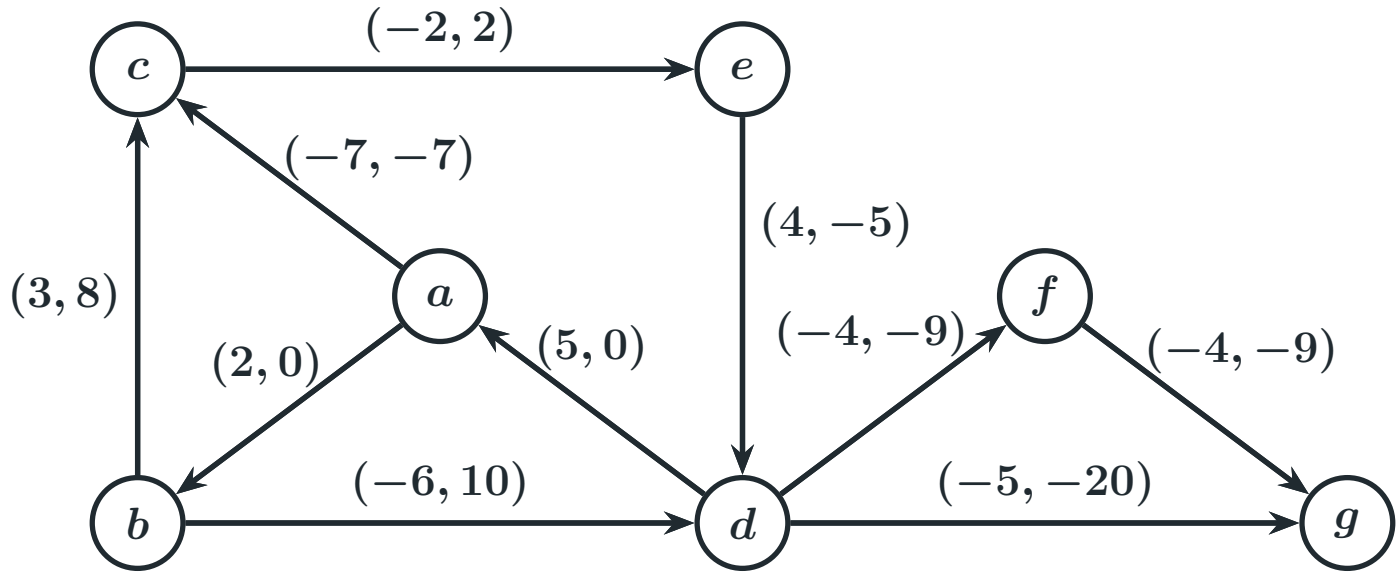

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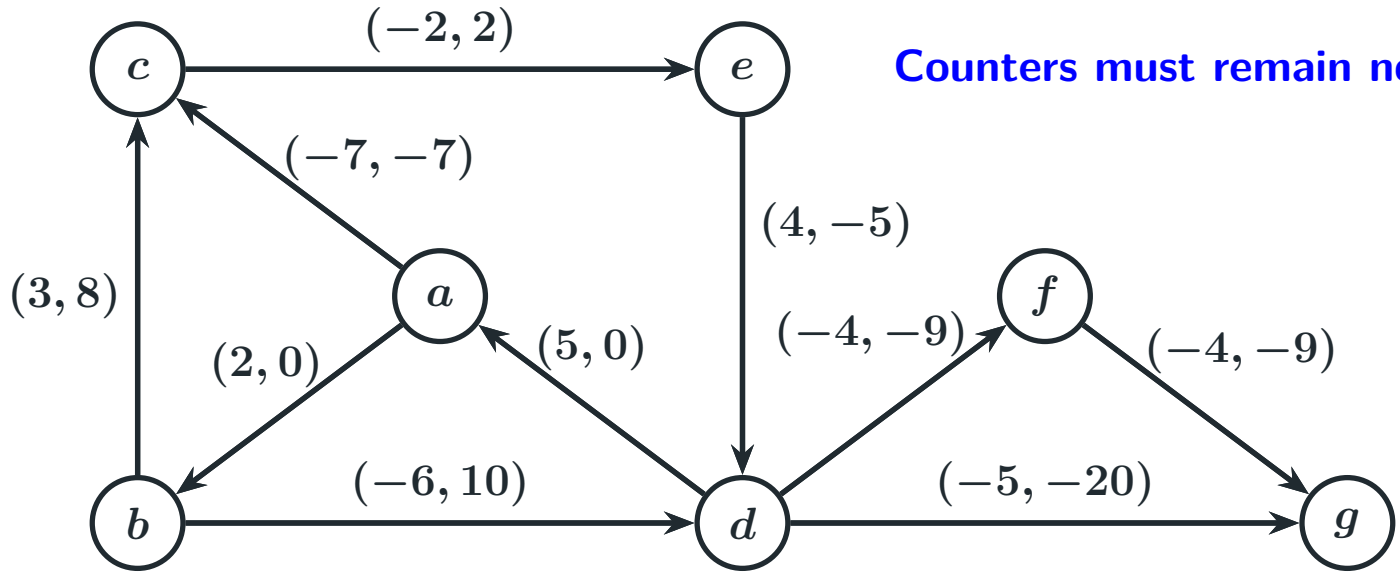

Counters must remain non-negative!

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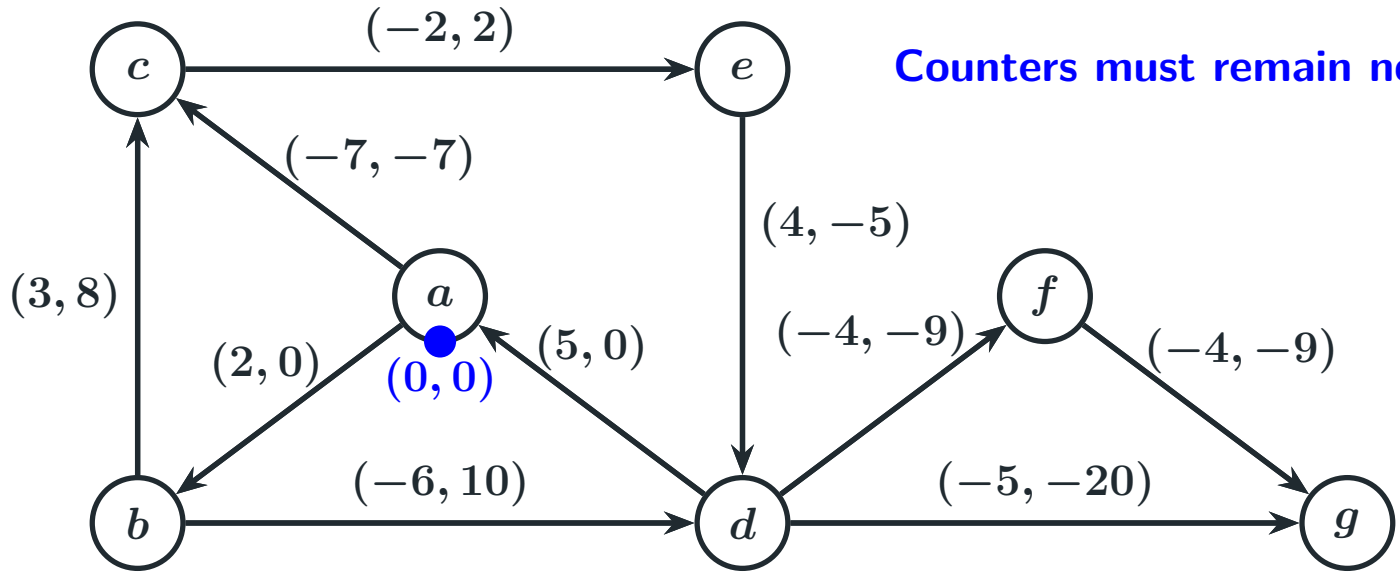

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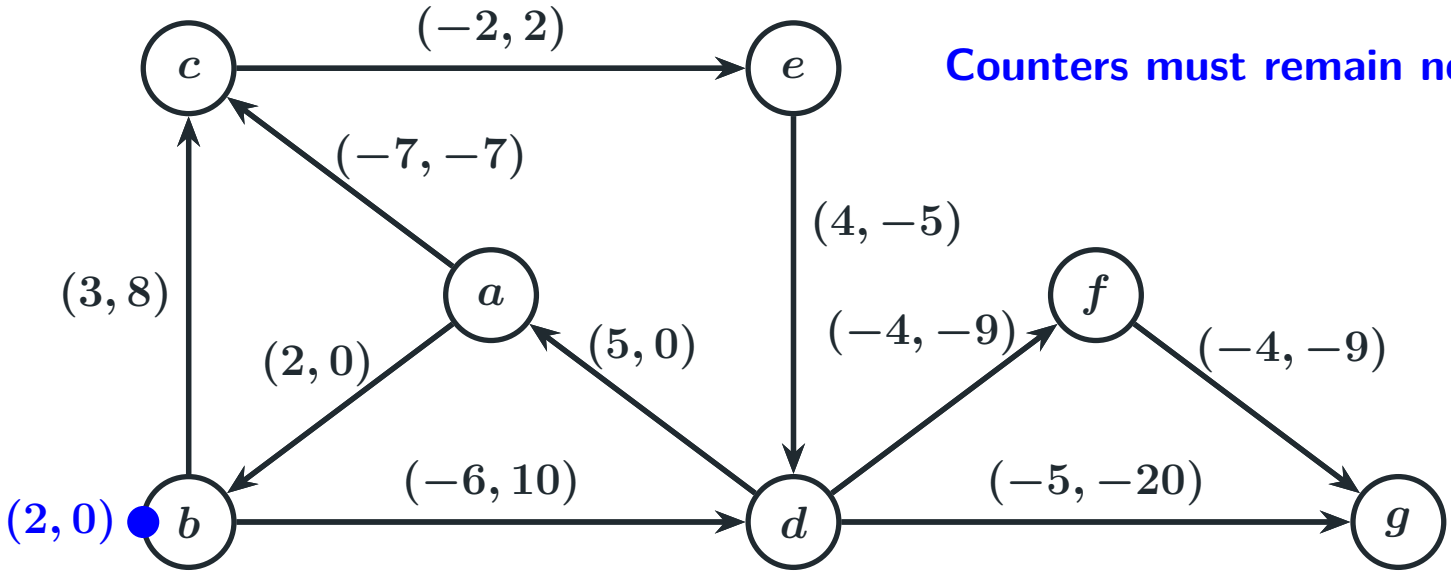

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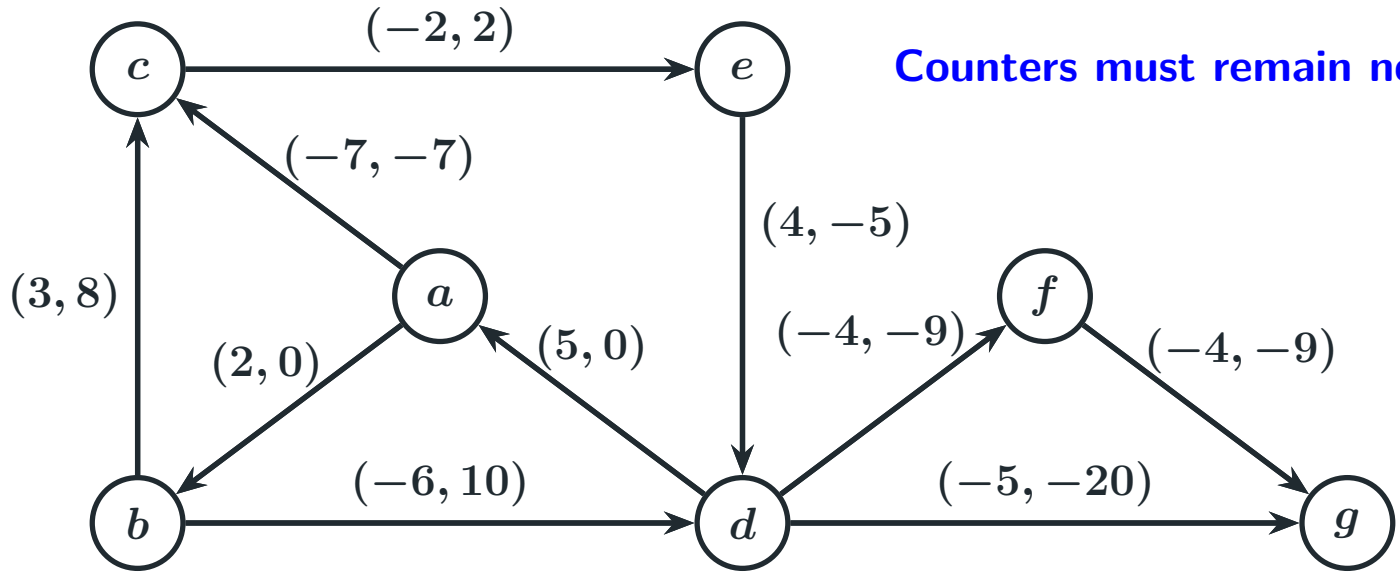

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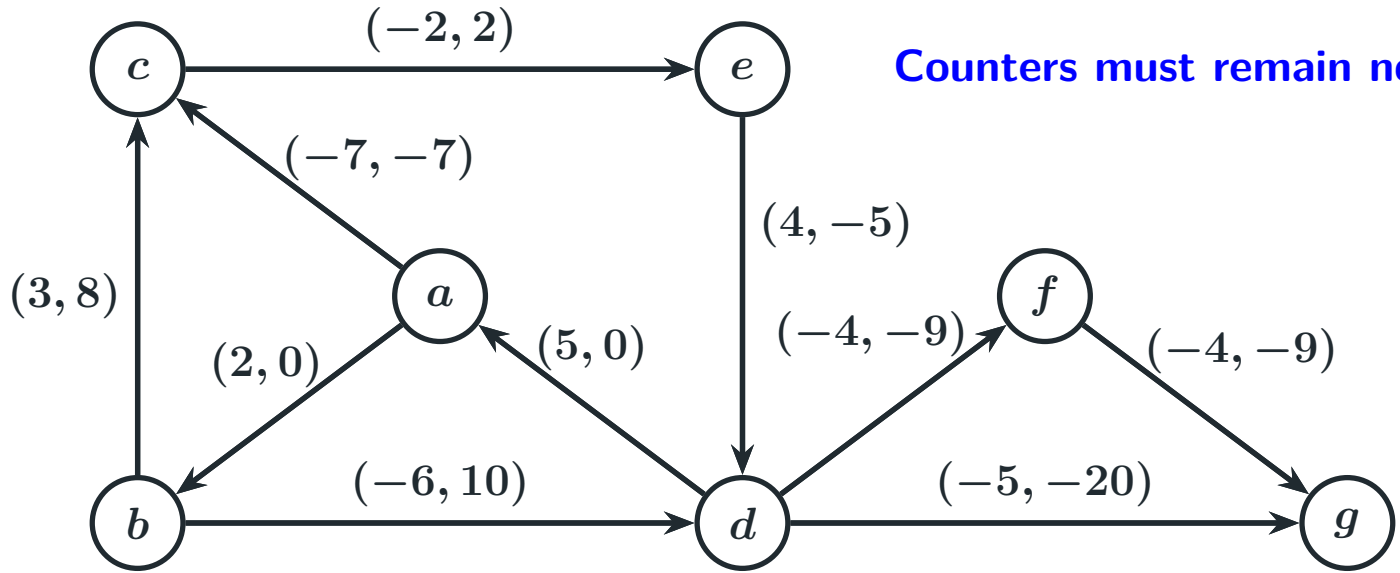

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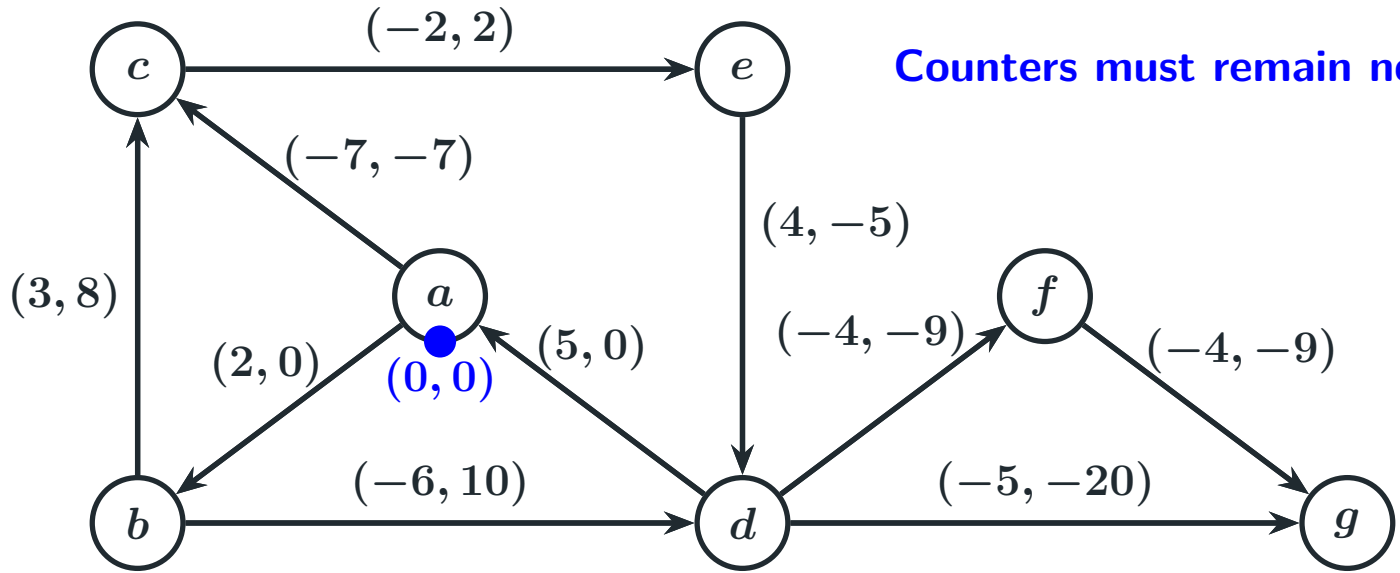

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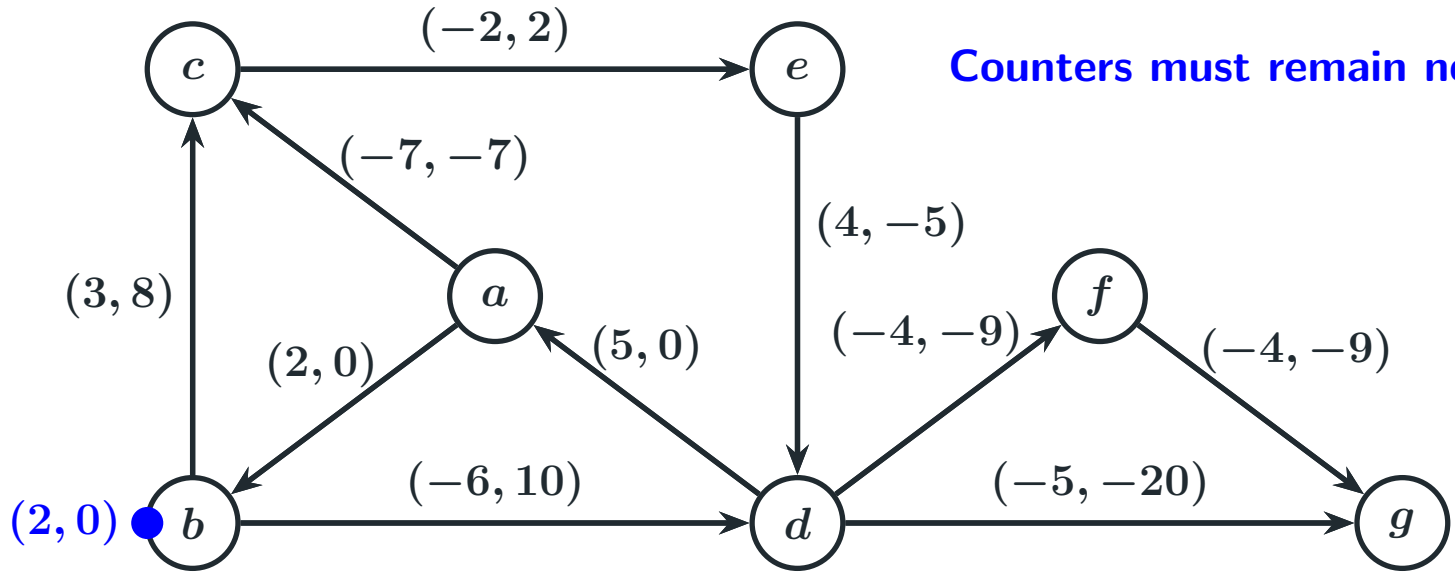

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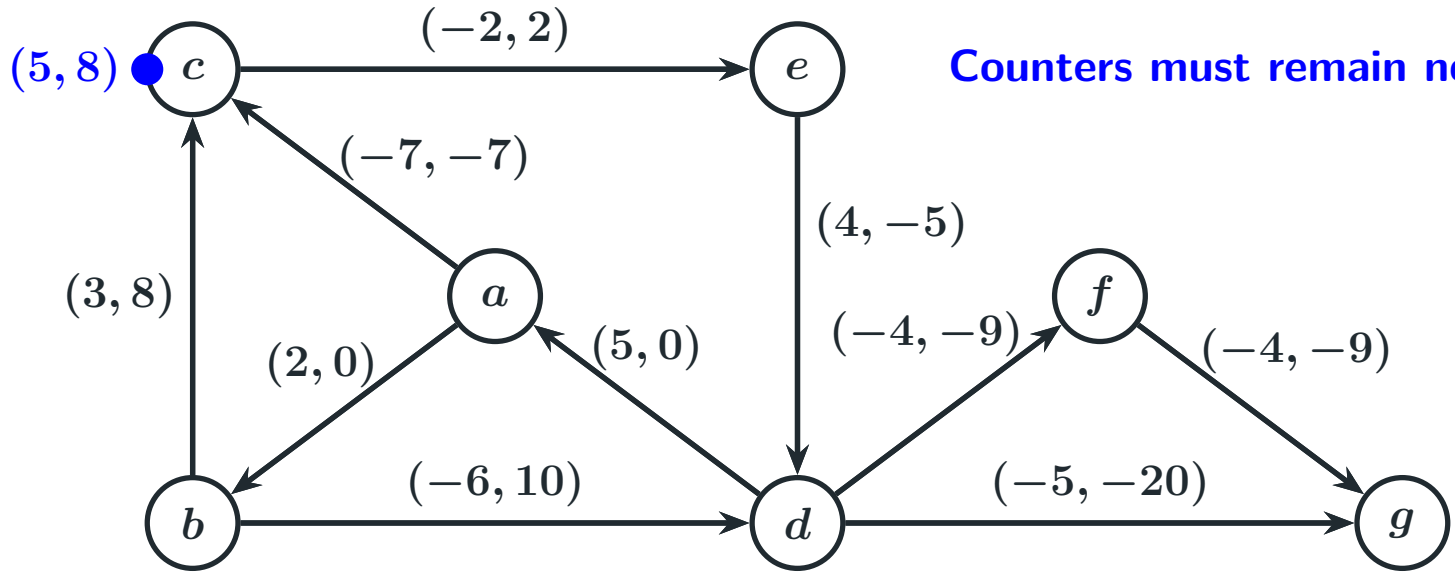

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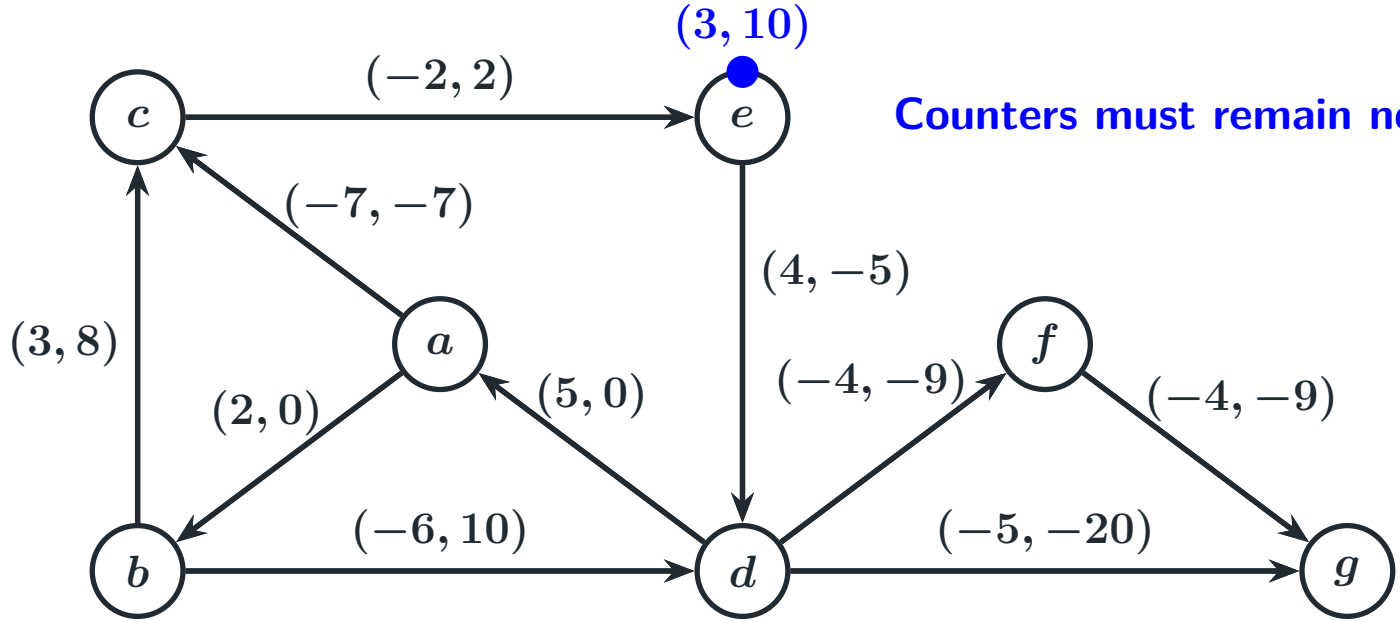

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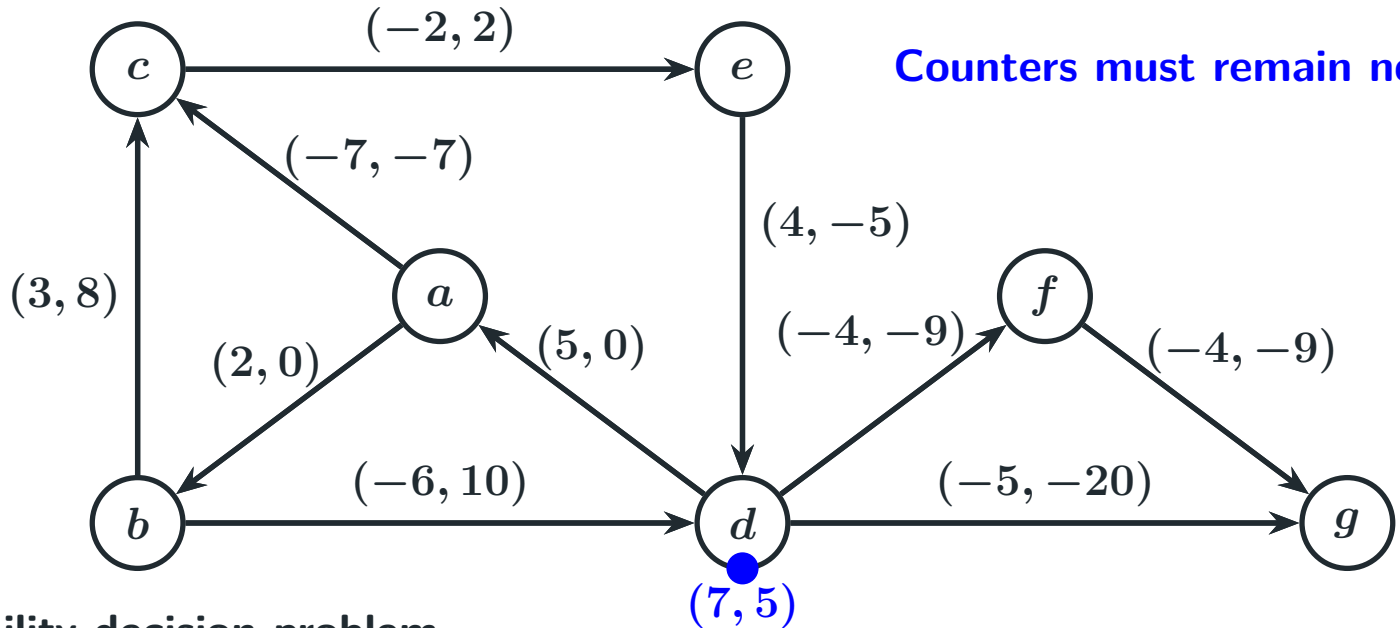

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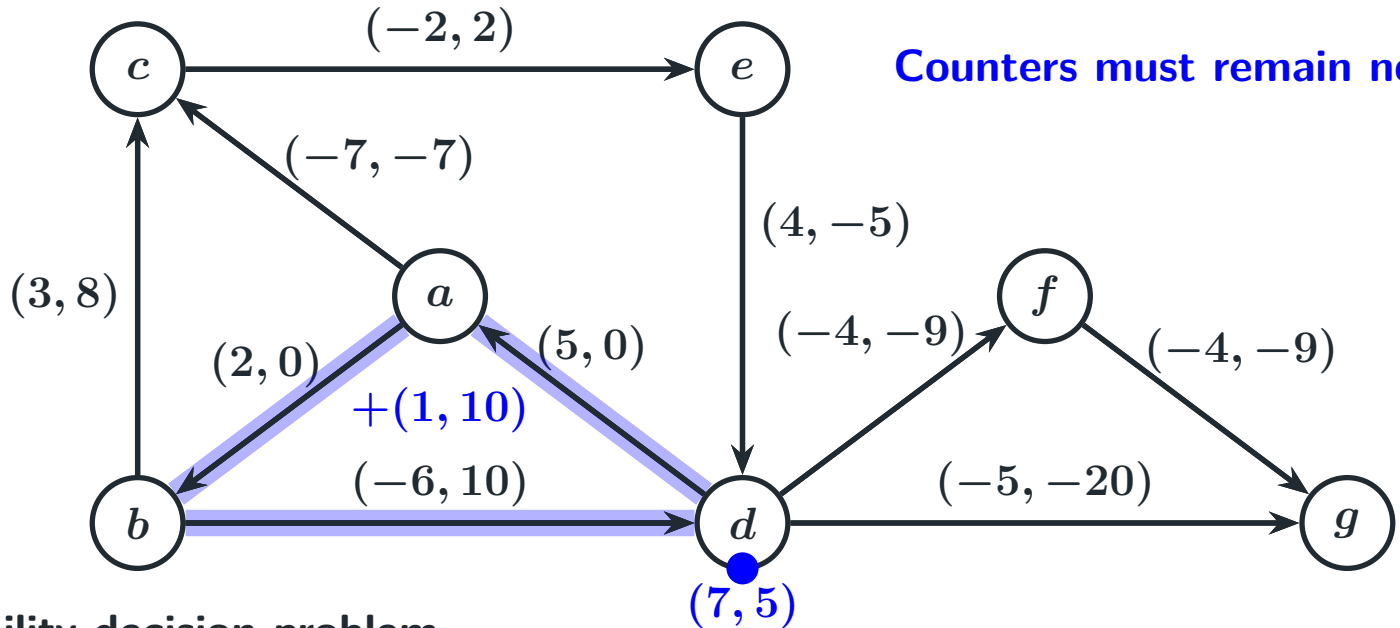

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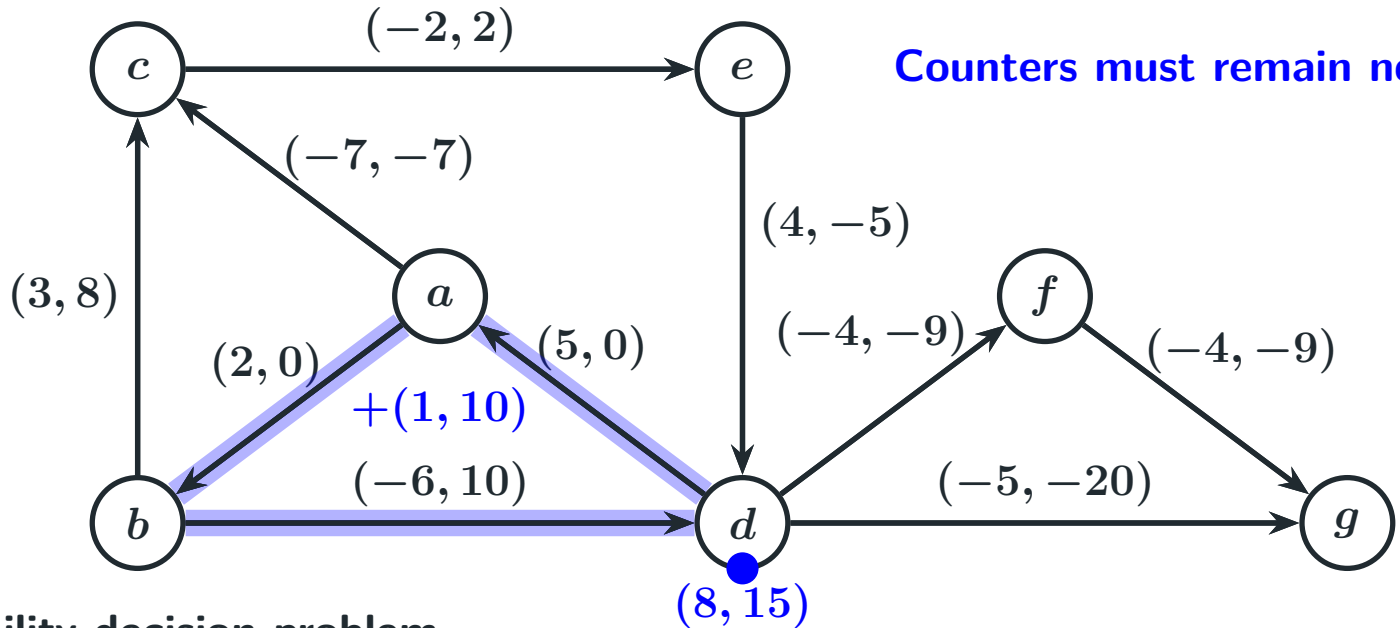

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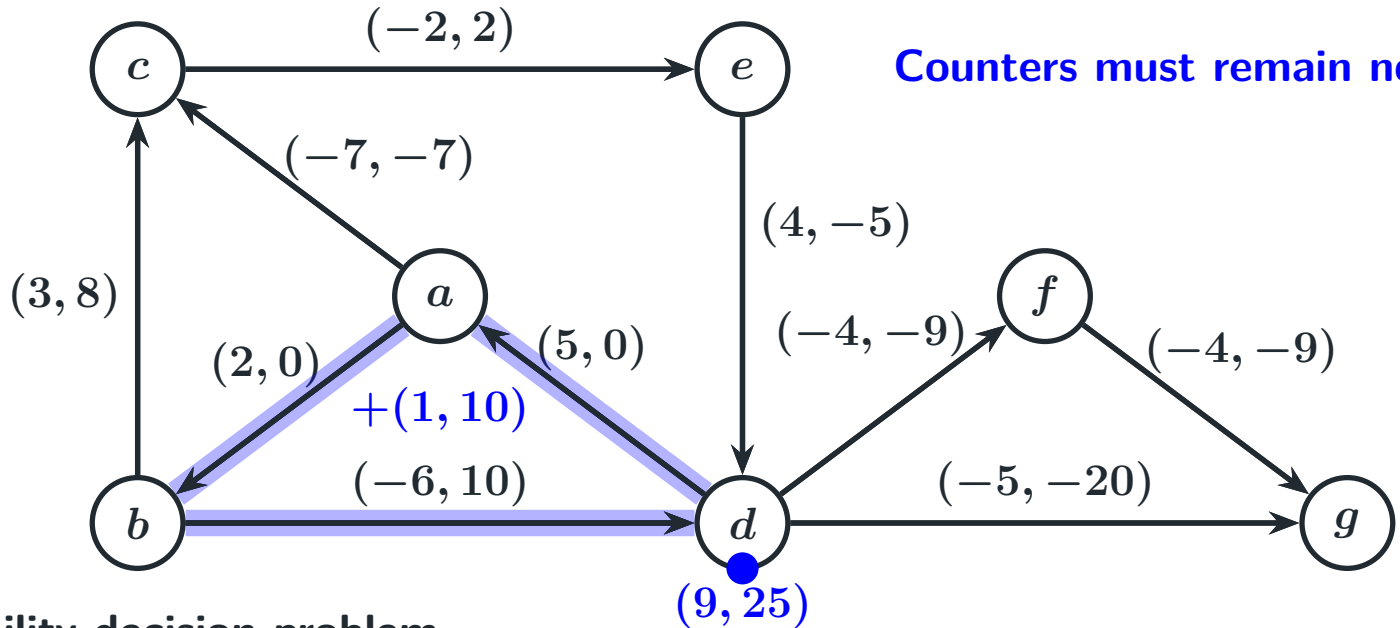

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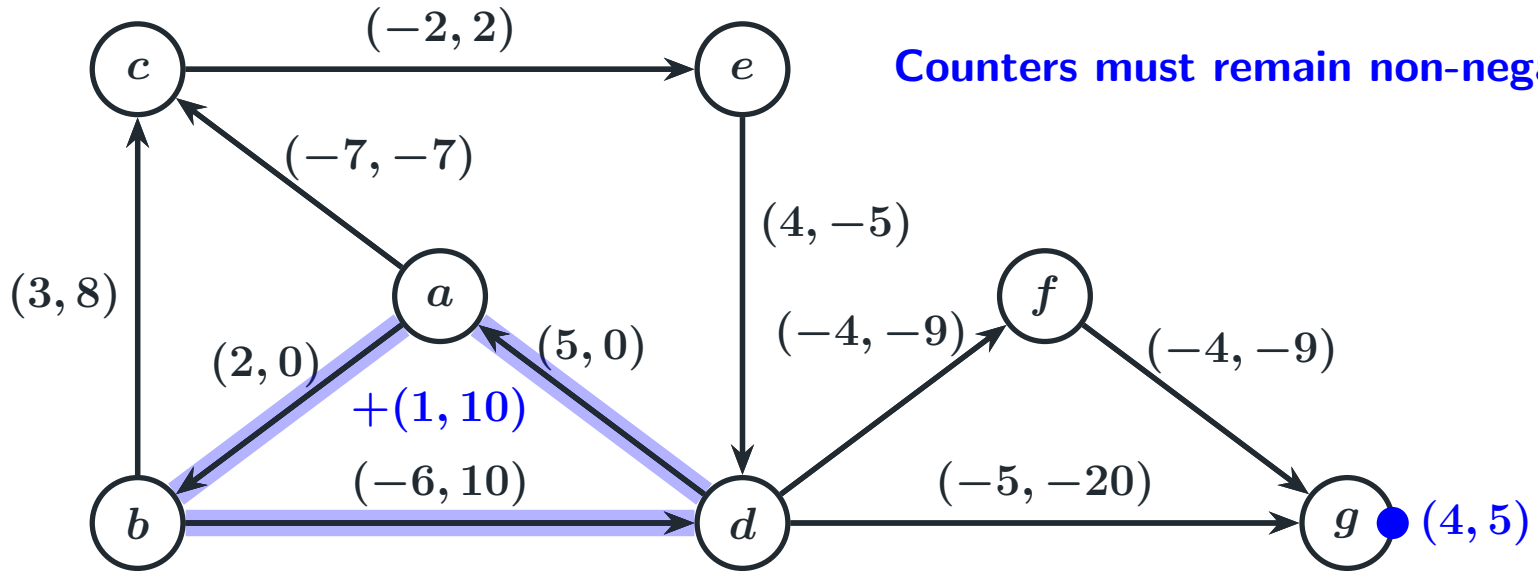

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For the remainer of this talk...

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n is the size of the instance in unary.

Preview of this talk

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Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

Coverability in d-VASS conditionally requires $n^{2^{\Omega(d)}}$ time.

Parameterised reduction from k-clique detection.

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Part 3: 1-VASS

Coverability in 1-VASS can be decided in n^2 time

Coverability in (binary) 1-VASS is in AC^1.

Open Problem

What is the *exact* complexity of of coverability in 1-VASS ?

The Complexity of Coverability in d-VASS

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Theorem. In a d-VASS, if there is a run from $(s, \vec{x})$ that covers $(t, \vec{y})$, then there is a run from $(s, \vec{x})$ that covers $(t, \vec{y})$ of length at most $n^{2^{\mathcal{O}(d)}}$. [Künnemann, Mazowiecki, Schütze, S-B, Węgrzycki '23]

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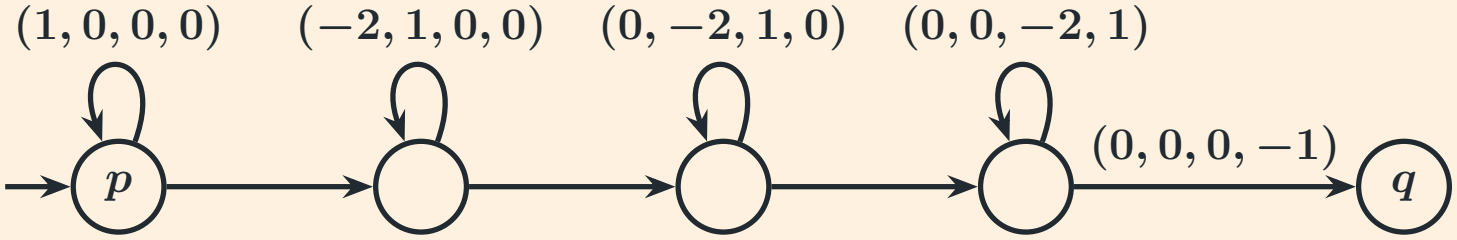

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does not directly imply :(

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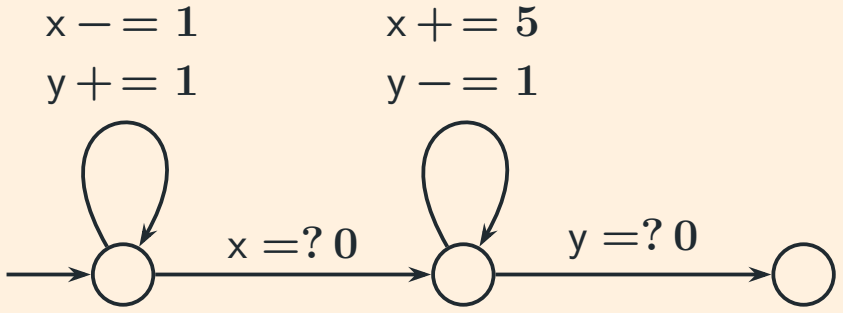

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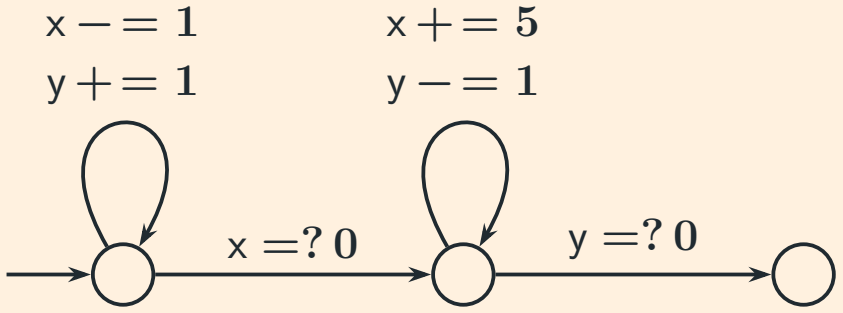

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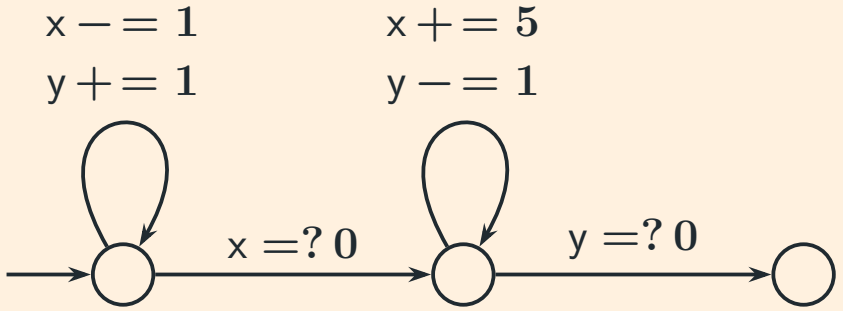
1. $\texttt{LOOP}(\mathsf{x} \mathrel{-}= 1, \mathsf{y} \mathrel{+}= 1)$
2. $\mathsf{x} =? 0$
3. $\texttt{LOOP}(\mathsf{x} \mathrel{+}= 5, \mathsf{y} \mathrel{-}= 1)$
4. $\mathsf{y} =? 0$

Post: $\mathsf{x} = x \cdot 5,\ \mathsf{y} = 0$



Proof Idea: Use Bounded Two-Counter Machines

Idea: Reduce detecting a 2^d-clique in a 2^d-partite n-vertex directed graph to coverability.

First, reduce to coverability in a $n^{2^{\mathcal{O}(d)}}$-bounded two-counter machine.

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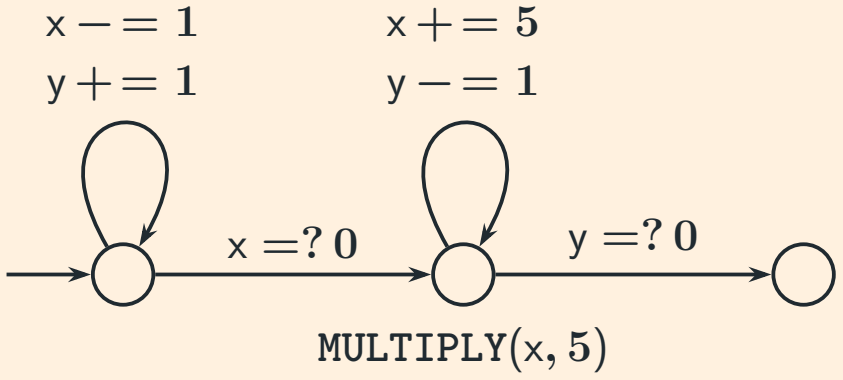

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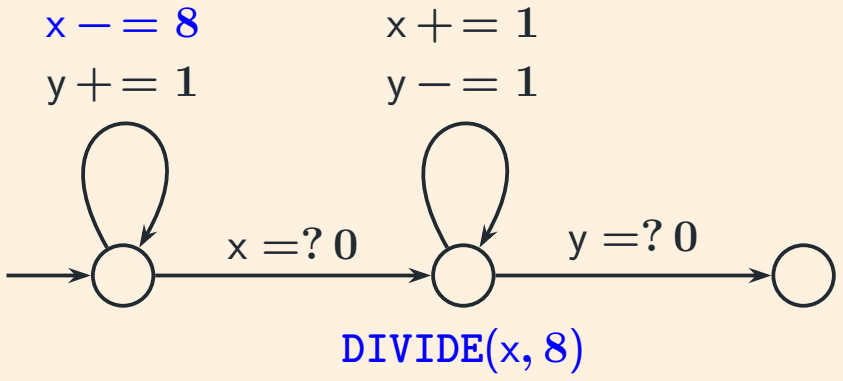
Pre: $\mathsf{x} = x,\ \mathsf{y} = 0$

1. $\texttt{LOOP}(\mathsf{x} \mathrel{-}= 8, \mathsf{y} \mathrel{+}= 1)$
2. $\mathsf{x} =? 0$
3. $\texttt{LOOP}(\mathsf{x} \mathrel{+}= 1, \mathsf{y} \mathrel{-}= 1)$
4. $\mathsf{y} =? 0$

Post: $\mathsf{x} = x \div 8, \mathsf{y} = 0$



Detecting Cliques using Divisibility Tests

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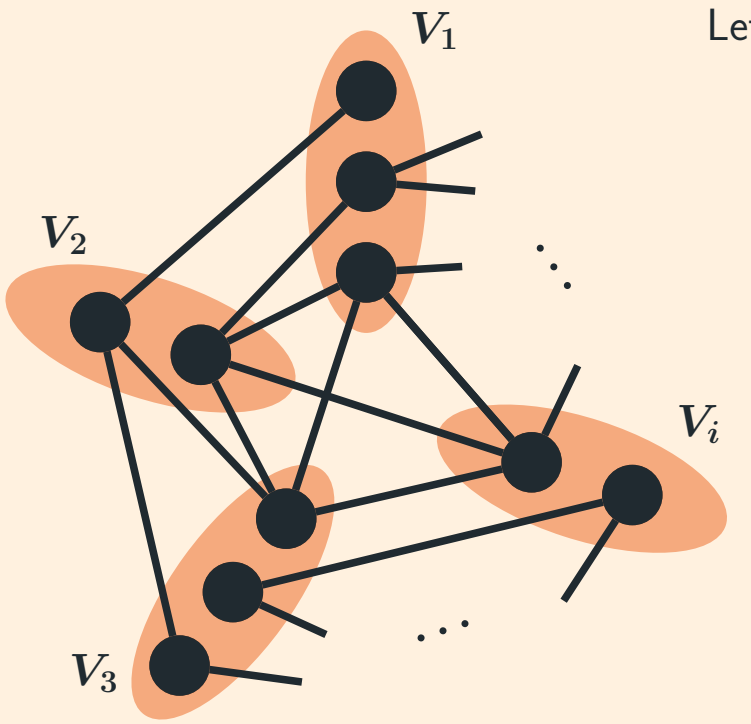
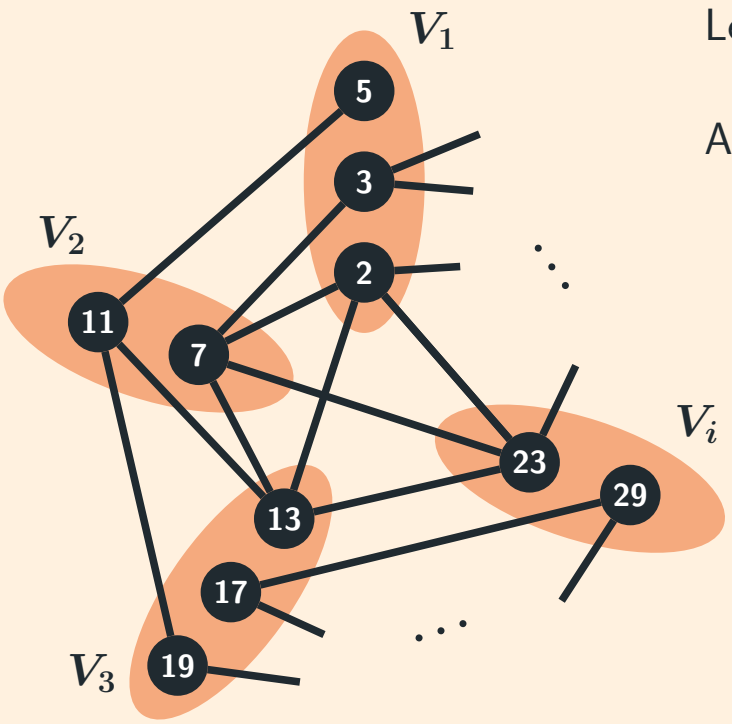


Let $(V_1 \cup V_2 \cup \cdots \cup V_k, E)$ be a k-partite n-vertex graph.

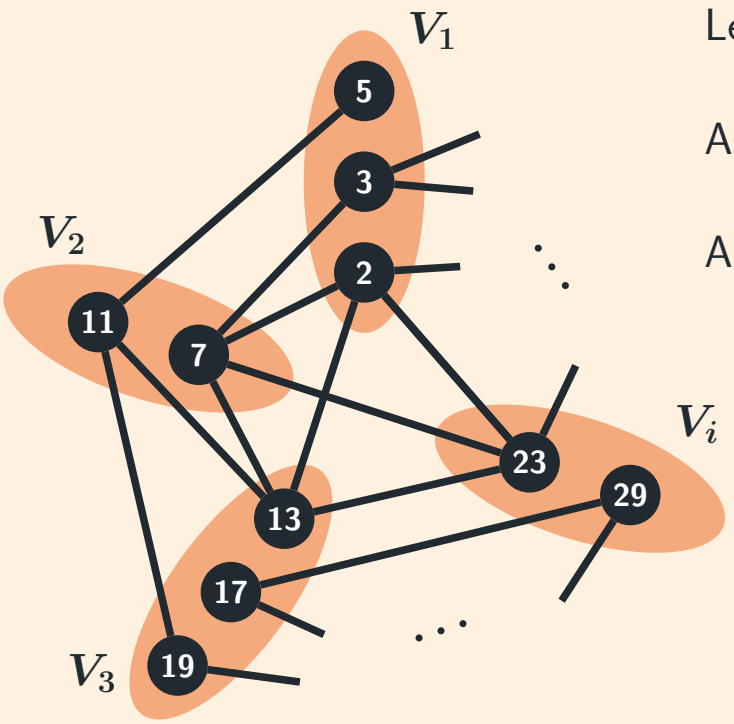
Detecting Cliques using Divisibility Tests



Let $(V_1 \cup V_2 \cup \cdots \cup V_k, E)$ be a k-partite n-vertex graph.

Associate the first n primes with the verticies.

Detecting Cliques using Divisibility Tests



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A candidate k-clique is represented by a product of k primes.

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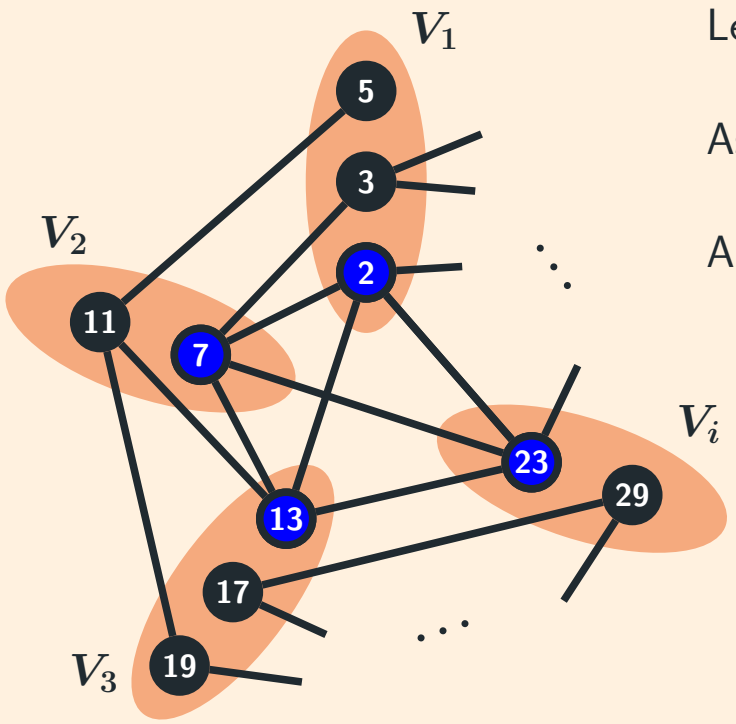

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Example: $c = 2 \cdot 7 \cdot 13 \cdot \ldots \cdot 23$.

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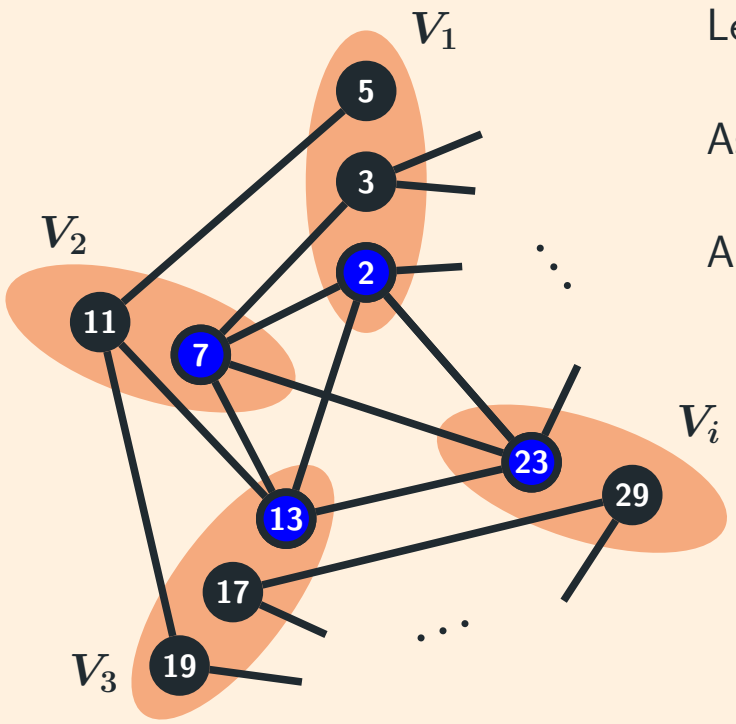

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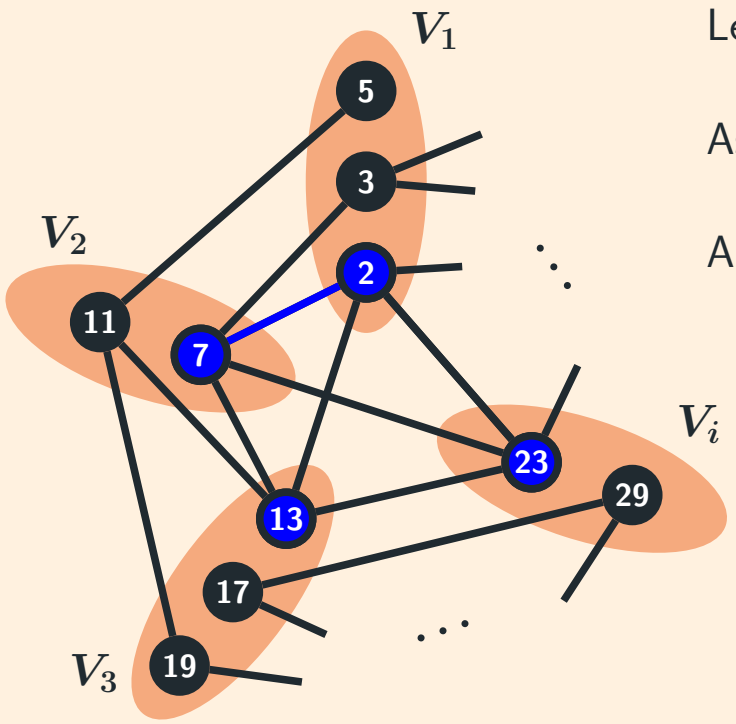

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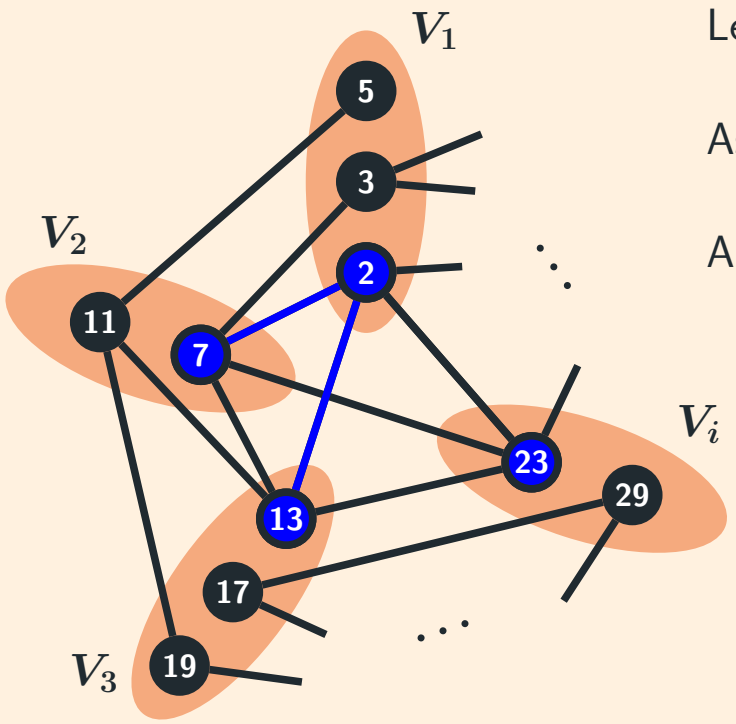
Example: $(2 \cdot 7)|c\,?$

Detecting Cliques using Divisibility Tests



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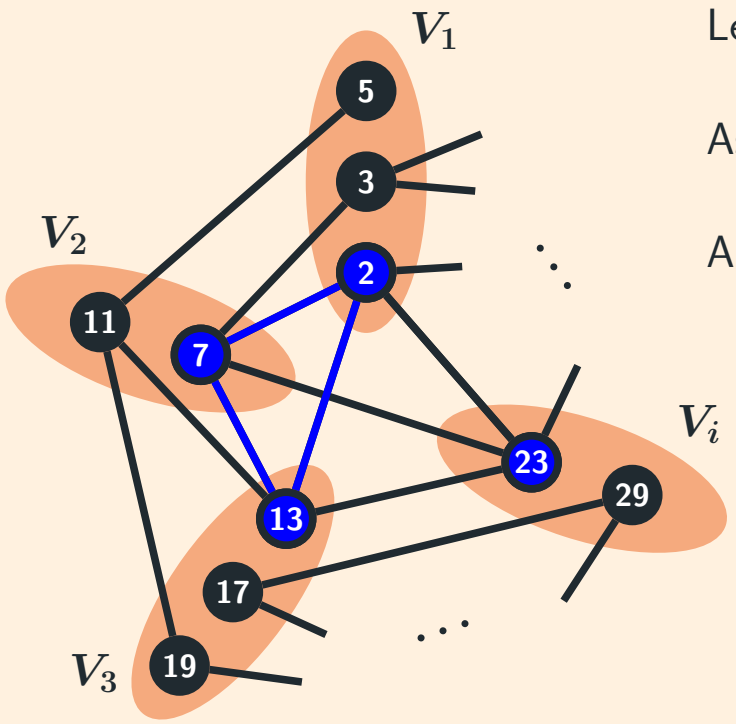
Detecting Cliques using Divisibility Tests



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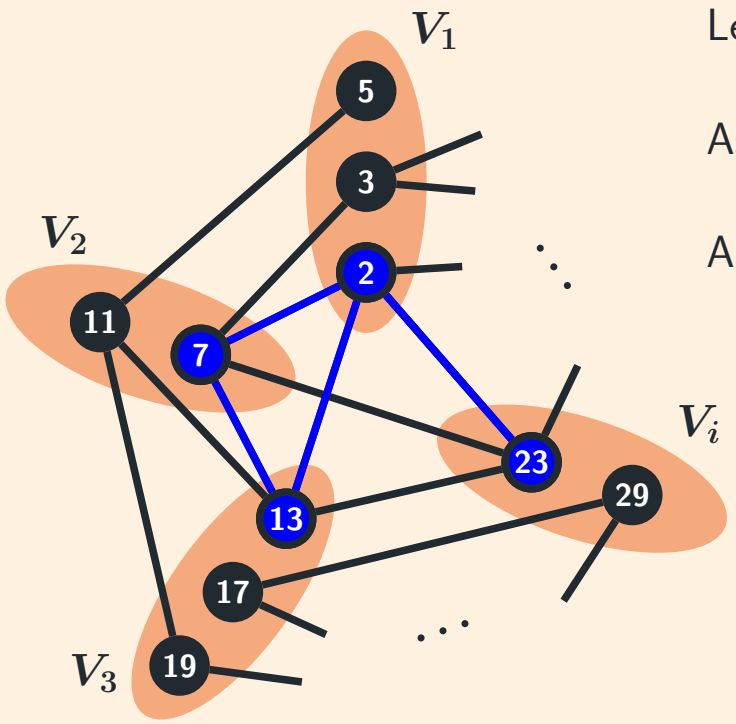

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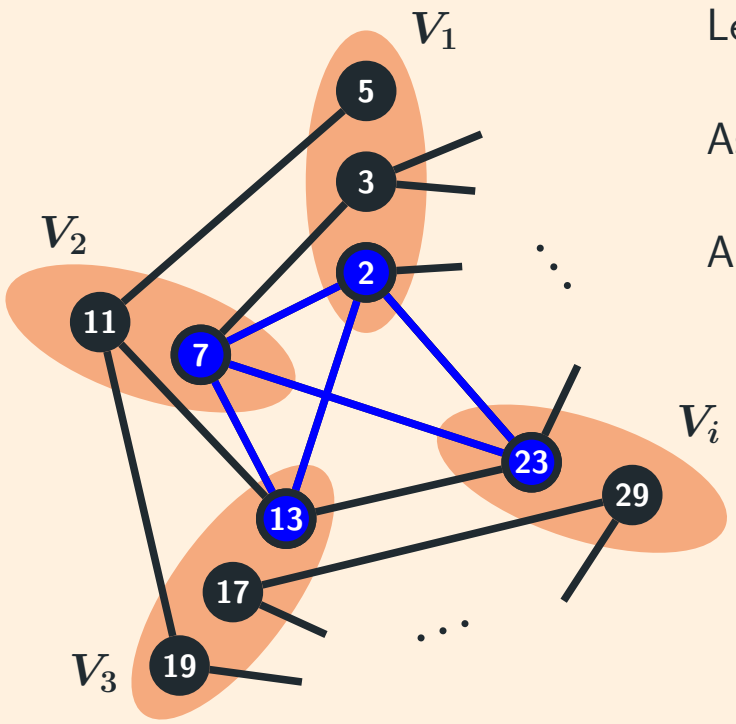
Detecting Cliques using Divisibility Tests



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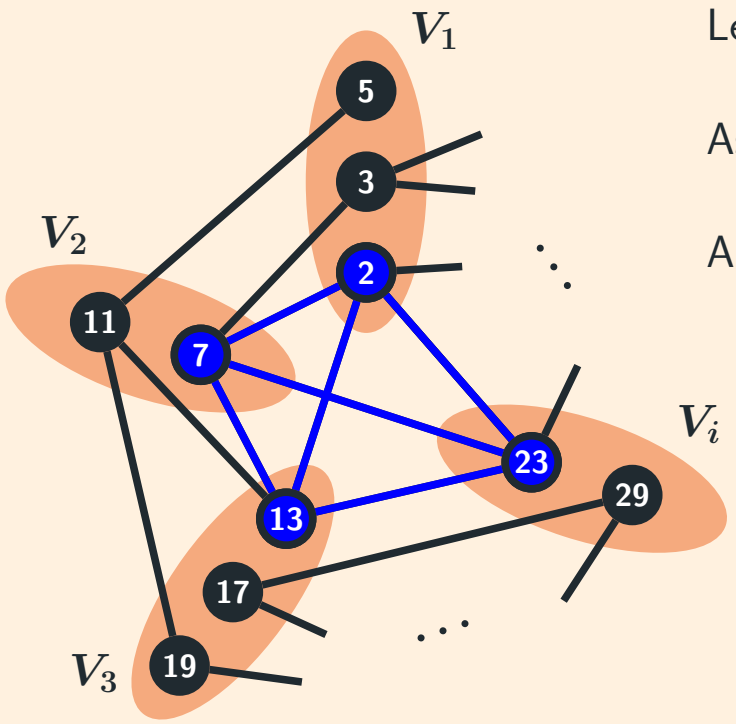

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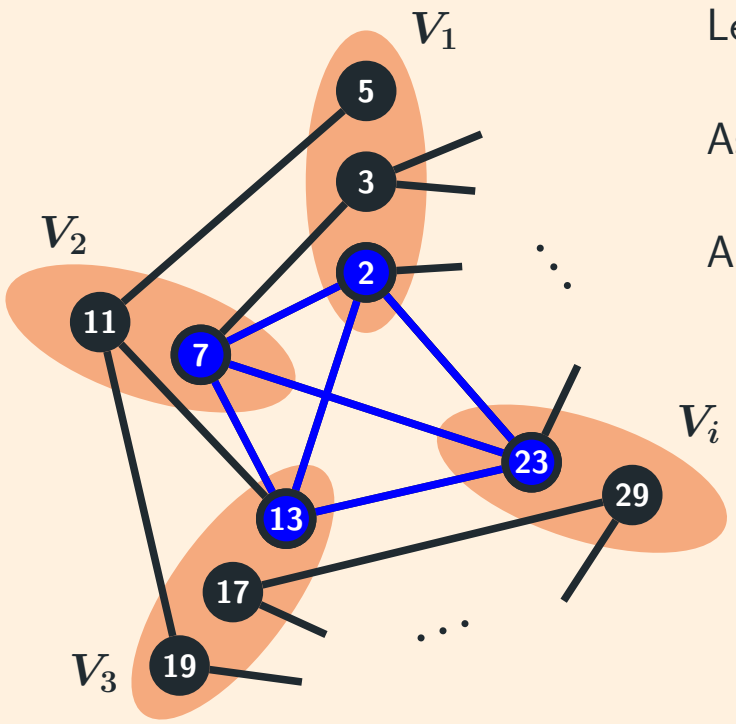

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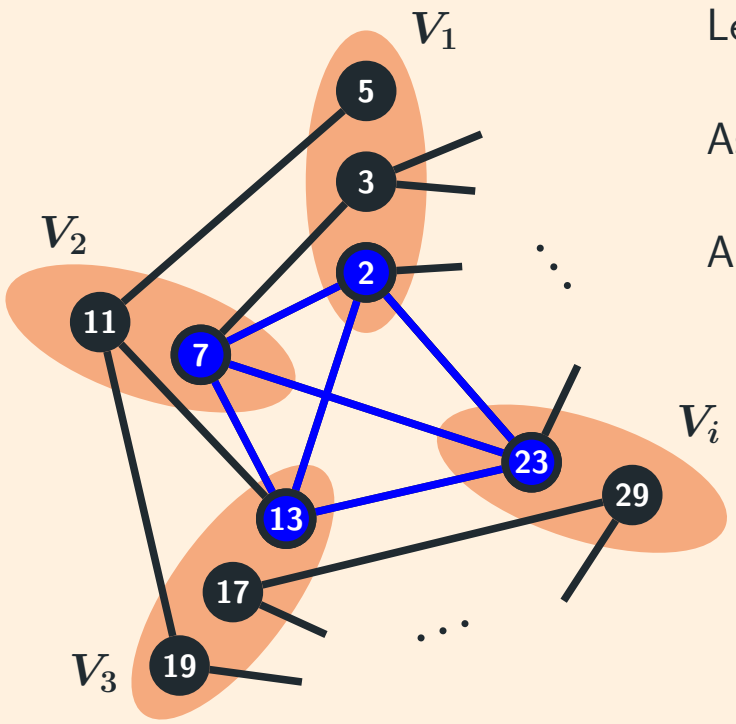
There exist $p_1 \in \text{Primes}(V_1), \ldots, p_k \in \text{Primes}(V_k)$

Detecting Cliques using Divisibility Tests



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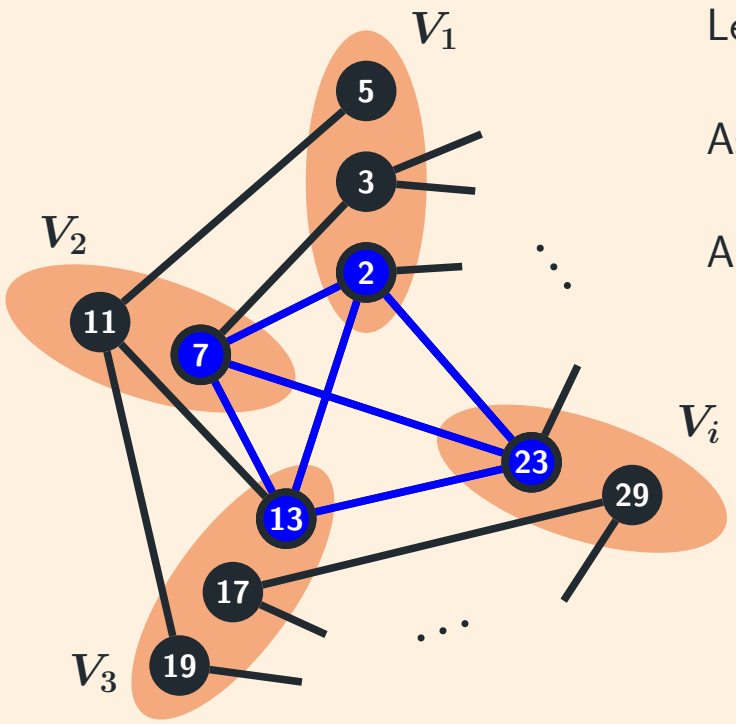

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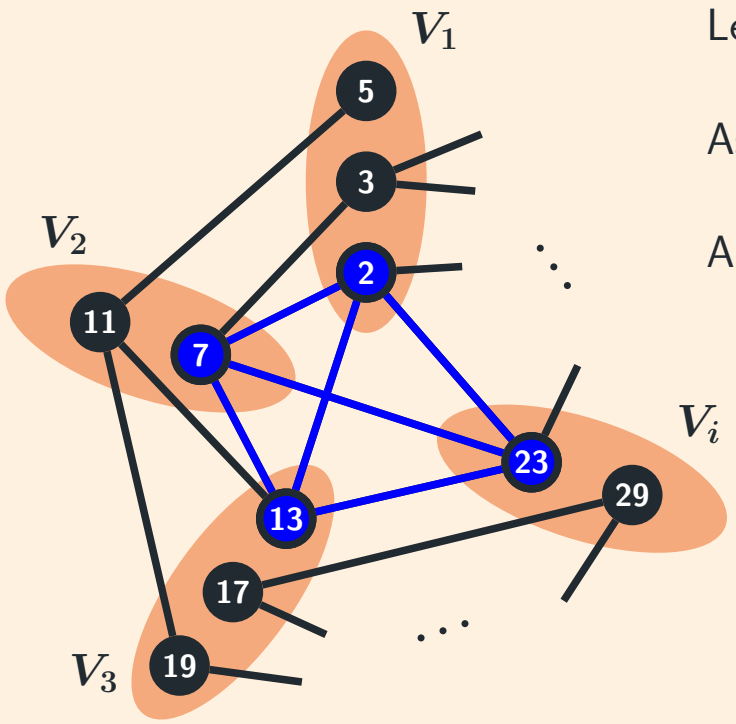

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Bounded Two-Counter Machine Implementation

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Bounded Two-Counter Machine Implementation

There e k, there

is an ec k-clique.

Guessing with Nondeterministic Branching

Bounded Two-Counter Machine Implementation

There e… k, there is an e… k-clique.

Guessing with Nondeterministic Branching

Pre: $\mathsf{x} = x$

1. $\texttt{GUESS}\ c \in \{1, 2, 3\}$
2. $\mathsf{x} \mathrel{+}= c$

Bounded Two-Counter Machine Implementation

There e… k, there is an e… k-clique.

Guessing with Nondeterministic Branching

Pre: $x = x$

1. GUESS $c \in \{1, 2, 3\}$
2. $x \mathrel{+}= c$

Post: $x = x + 1$, or
$x = x + 2$, or
$x = x + 3$.

Bounded Two-Counter Machine Implementation

There e… k, there is an e… k-clique.

Guessing with Nondeterministic Branching

Pre: $x = x$

1. GUESS $c \in \{1, 2, 3\}$
2. $x += c$

Post: $x = x + 1$, or
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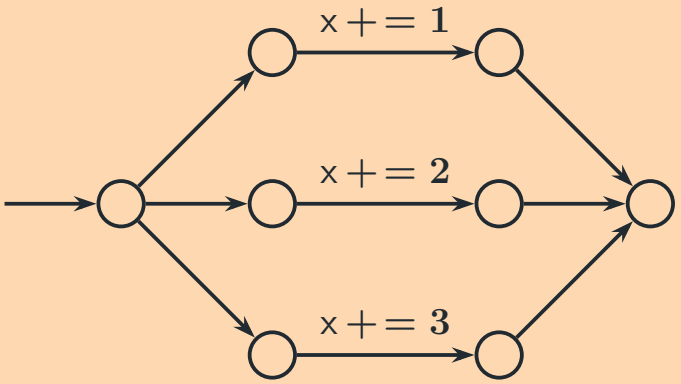

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Part one: Guess a candidate clique.

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Part one: Guess a candidate clique.

Pre: $\text{x} = 1,\ \text{y} = 0$.

`GUESS` $p_1 \in \text{Primes}(V_1)$

`MULTIPLY`(x, p_1)

Bounded Two-Counter Machine Implementation

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`MULTIPLY`(x, p_k)

Post: $\text{x} = p_1 \cdot \ldots \cdot p_k,\ \text{y} = 0$.

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- `MULTIPLY`(x, p_1)
- $\vdots$
- `GUESS` $p_k \in \text{Primes}(V_k)$
- `MULTIPLY`(x, p_k)

Post: $x = p_1 \cdot \ldots \cdot p_k,\ y = 0$.

Part two: Check the candidate is a clique.

Bounded Two-Counter Machine Implementation

There exist $p_1 \in \text{Primes}(V_1), \ldots, p_k \in \text{Primes}(V_k)$ such that for every pair $1 \leq i < j \leq k$, there is an edge $\{p, q\} \in (V_i \times V_j) \cap E$ such that $(p \cdot q) \mid p_1 \cdot \ldots \cdot p_k \iff$ there exists a k-clique.

Part one: Guess a candidate clique.

Pre: $x = 1,\ y = 0$.

- GUESS $p_1 \in \text{Primes}(V_1)$
- MULTIPLY(x, p_1)
- $\vdots$
- GUESS $p_k \in \text{Primes}(V_k)$
- MULTIPLY(x, p_k)

Post: $x = p_1 \cdot \ldots \cdot p_k,\ y = 0$.

Part two: Check the candidate is a clique.

Pre: $x = p_1 \cdot \ldots \cdot p_k,\ y = 0$.

GUESS $\{p_1, p_2\} \in (V_1 \times V_2) \cap E$

Bounded Two-Counter Machine Implementation

There exist $p_1 \in \text{Primes}(V_1), \ldots, p_k \in \text{Primes}(V_k)$ such that for every pair $1 \leq i < j \leq k$, there is an edge $\{p, q\} \in (V_i \times V_j) \cap E$ such that $(p \cdot q) \mid p_1 \cdot \ldots \cdot p_k \iff$ there exists a k-clique.

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- $\texttt{GUESS}\ p_1 \in \text{Primes}(V_1)$
- $\texttt{MULTIPLY}(\text{x}, p_1)$
- $\vdots$
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- $\texttt{MULTIPLY}(\text{x}, p_k)$

Post: $\text{x} = p_1 \cdot \ldots \cdot p_k, \ \text{y} = 0$.

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Pre: $\text{x} = p_1 \cdot \ldots \cdot p_k, \ \text{y} = 0$.

- $\texttt{GUESS}\ \{p_1, p_2\} \in (V_1 \times V_2) \cap E$
- $\texttt{DIVIDE}(\text{x}, p_1 \cdot p_2)$

Bounded Two-Counter Machine Implementation

There exist $p_1 \in \text{Primes}(V_1), \ldots, p_k \in \text{Primes}(V_k)$ such that for every pair $1 \leq i < j \leq k$, there is an edge $\{p, q\} \in (V_i \times V_j) \cap E$ such that $(p \cdot q) \mid p_1 \cdot \ldots \cdot p_k \iff$ there exists a k-clique.

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Lemma. In $\mathit{poly}(n)$ time, one can construct a $\mathcal{O}(\log(k))$-VASS that can *simulate* an $\mathcal{O}(n^k)$-bounded two-counter machine of $\mathit{poly}(n)$ size. [Rosier and Yen '85]

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Remark. Termination is coverability.

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"Can I get to the end of the program with any (at least zero) value on each of the counters?"

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Theorem. Assuming the exponential time hypothesis, coverability in VASS requires $n^{2^{\Omega(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Preview of this talk

Part 1: d-VASS

Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

Coverability in d-VASS conditionally requires $n^{2^{\Omega(d)}}$ time.

Parameterised reduction from k-clique detection.

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Linear-time reduction from k-cycle detection.

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Coverability in (binary) 1-VASS is in AC^1.

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Corollary. Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

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Theorem. Assuming the k-cycle hypothesis, there does not exist an $\varepsilon > 0$ and an $\mathcal{O}(n^{2-\varepsilon})$-time algorithm that decides coverability in 2-VASS. [Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

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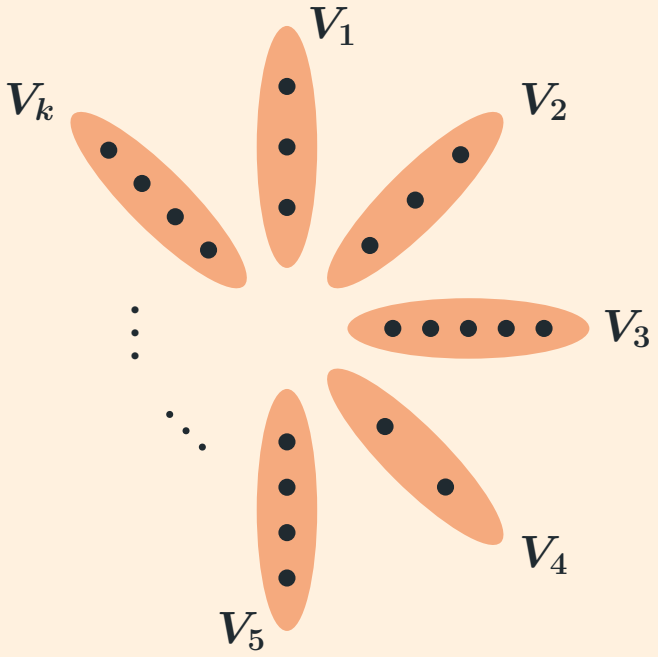

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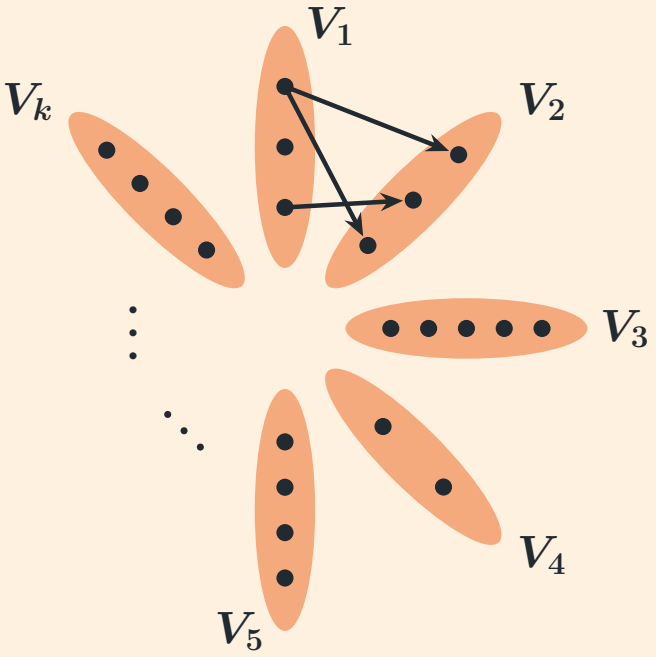

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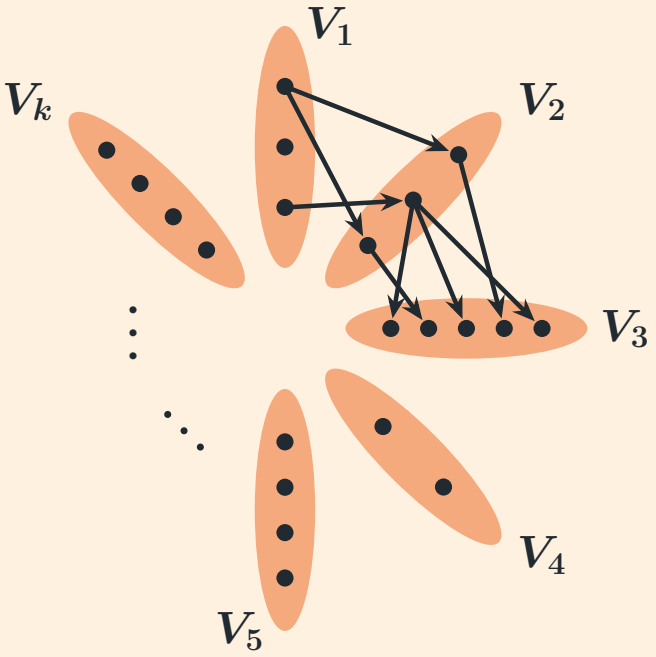

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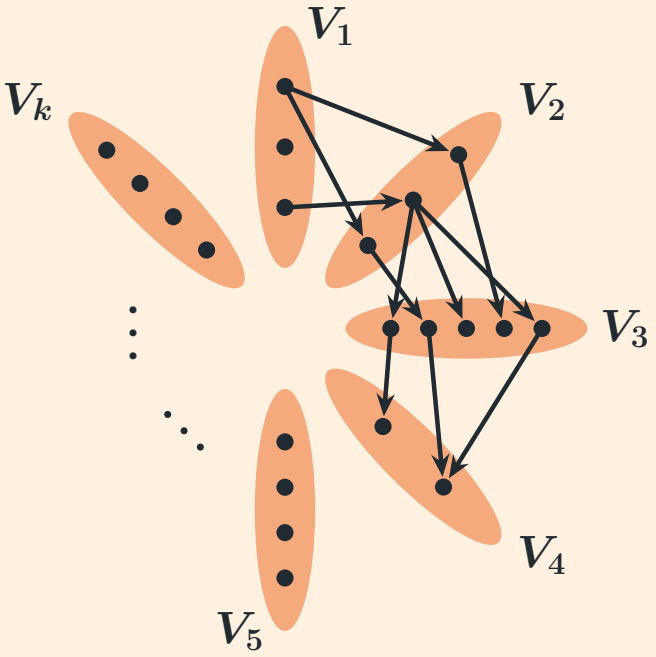

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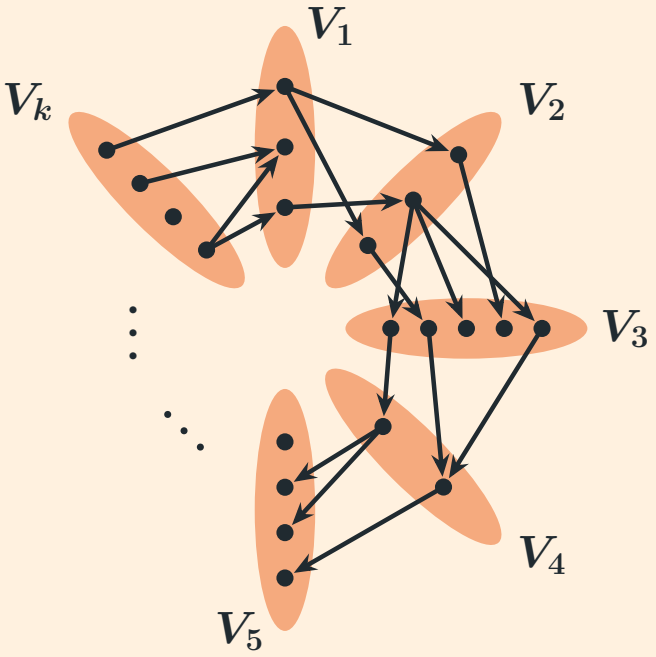

Reduction From k-Cycle Detection

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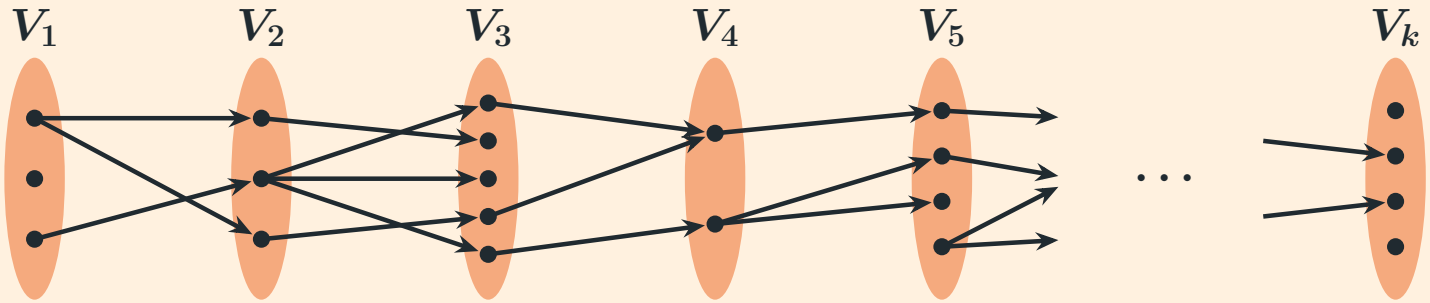

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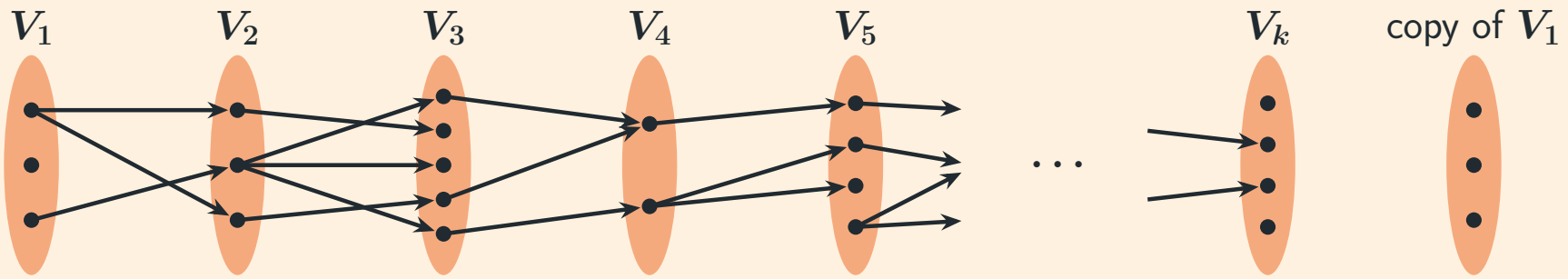

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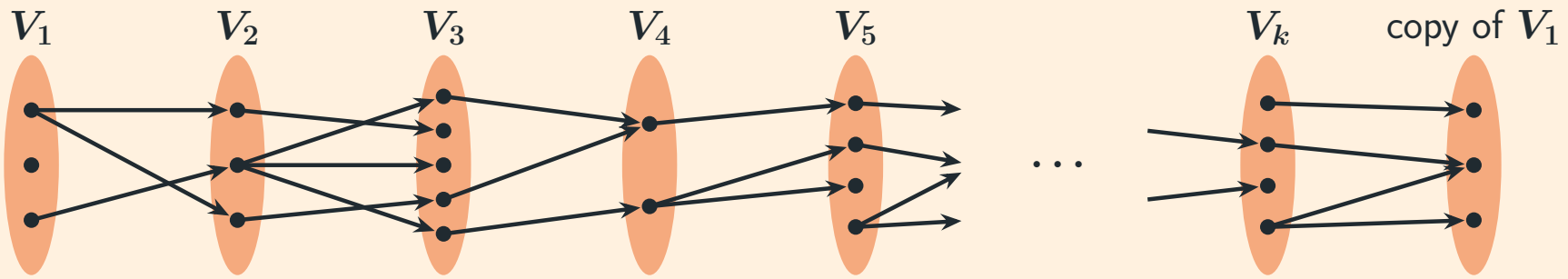

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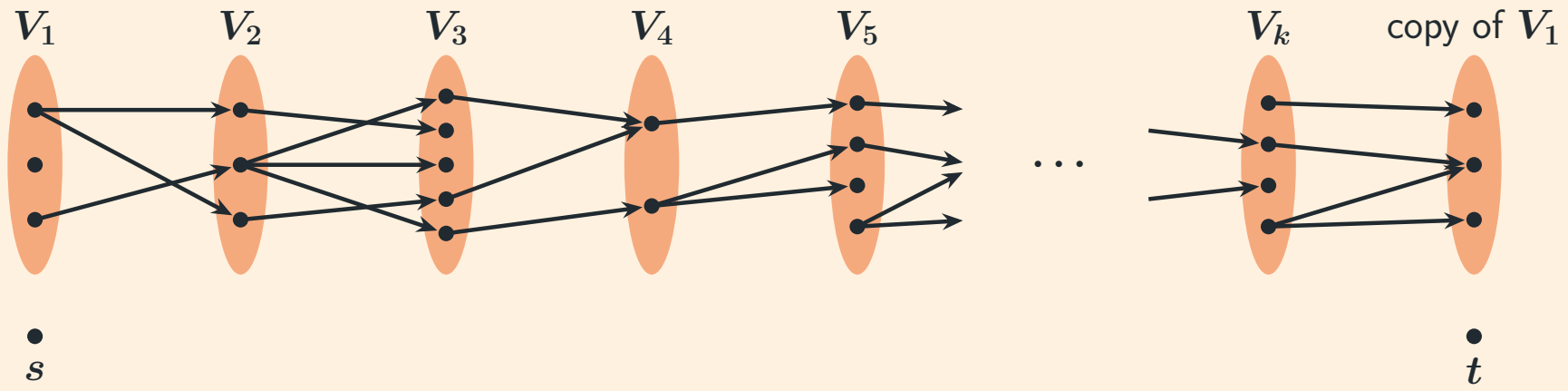

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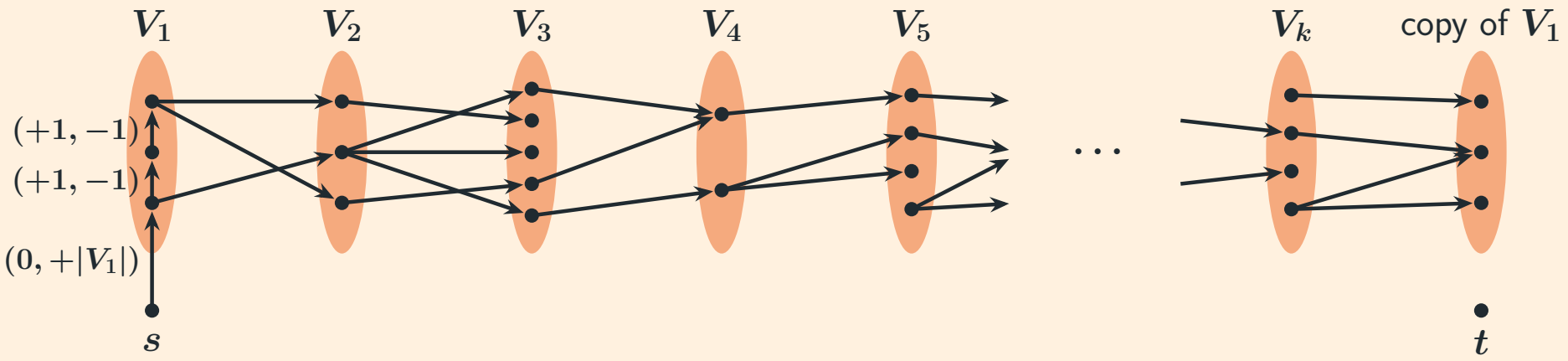

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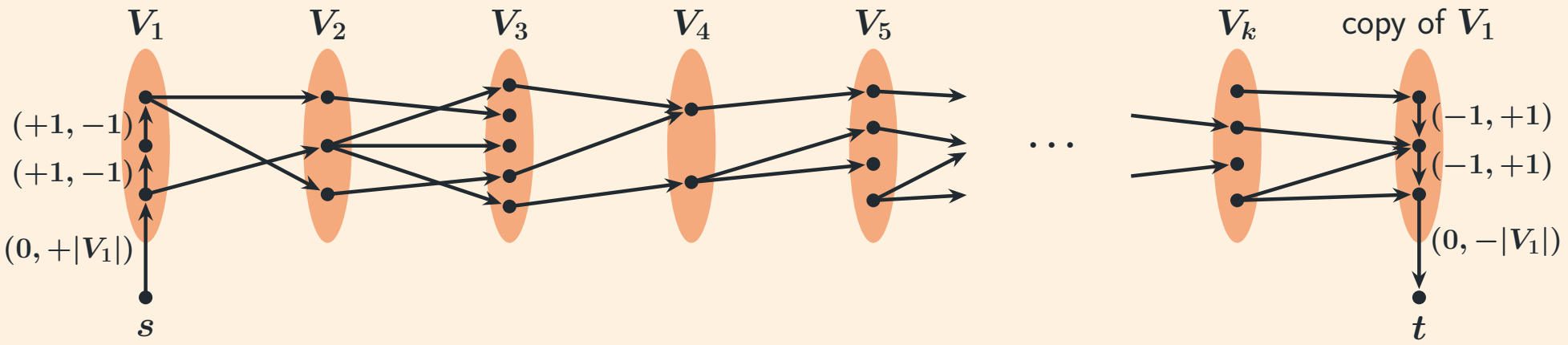

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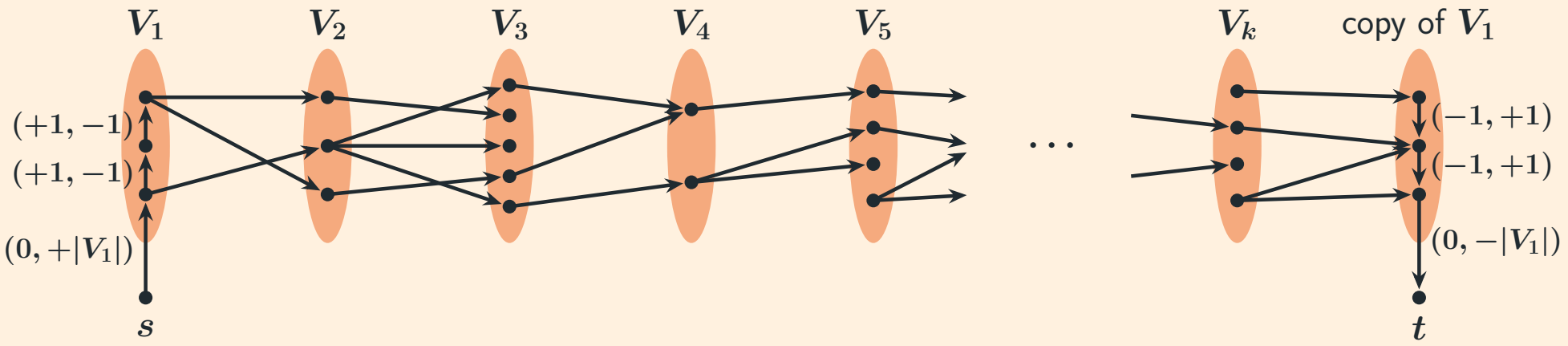

Claim. From $(s,(0,0))$ you can cover $(t,(0,0))$ $\iff$ there is a k-cycle the original graph.

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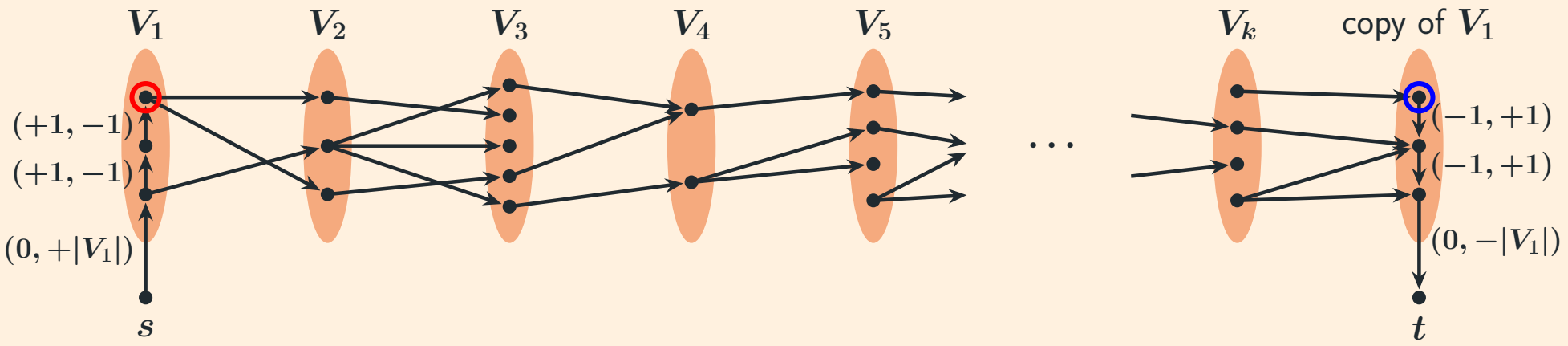

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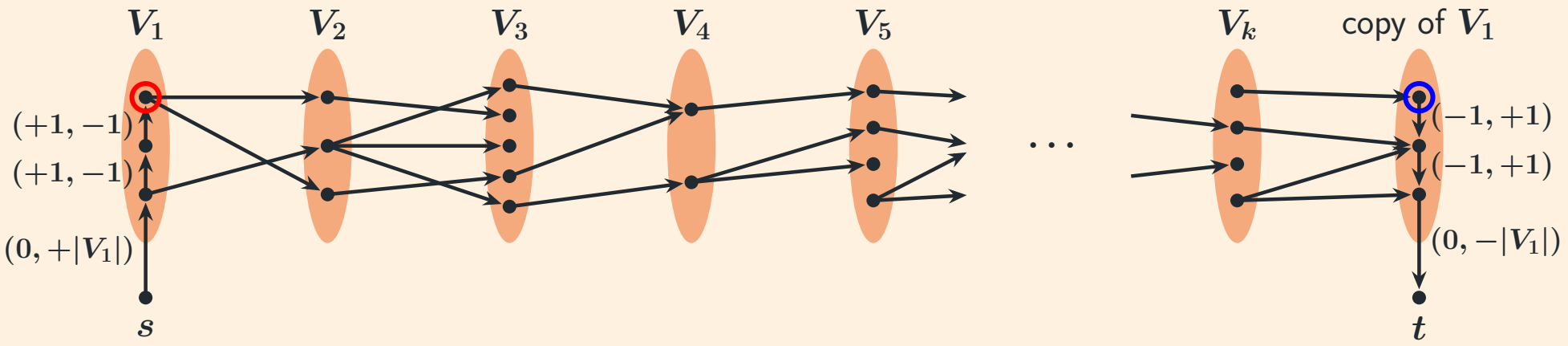

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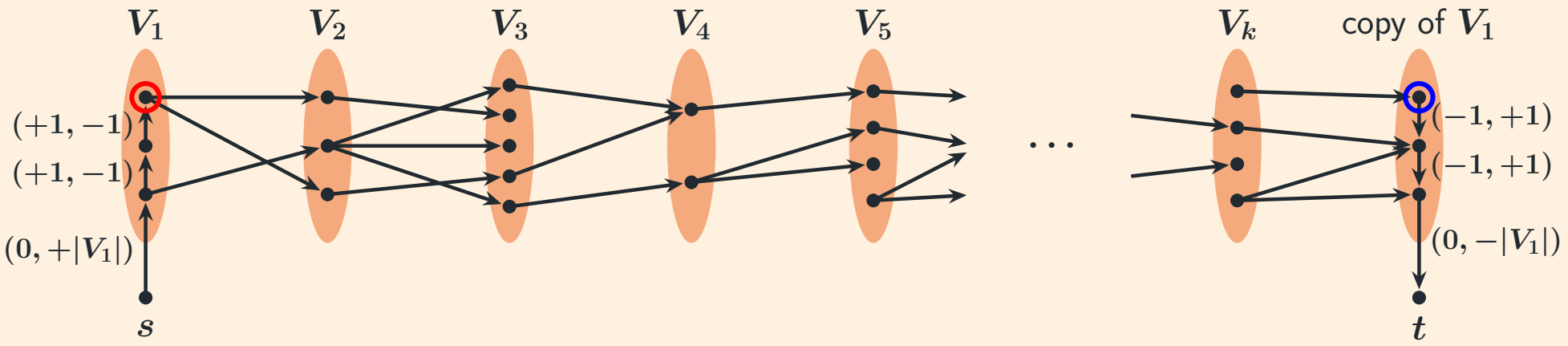

Claim. From $(s, (0, 0))$ you can cover $(t, (0, 0))$ $\iff$ there is a k-cycle the original graph.

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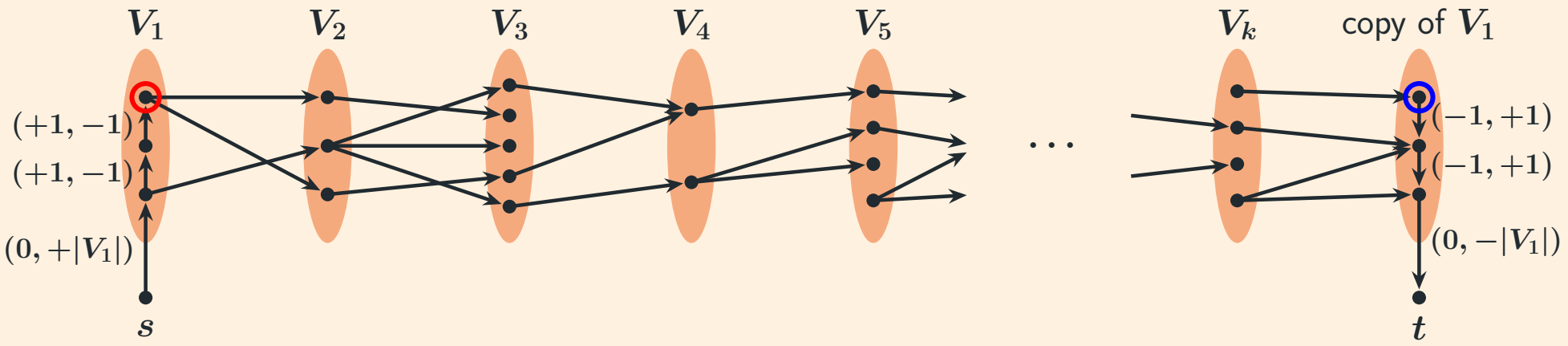

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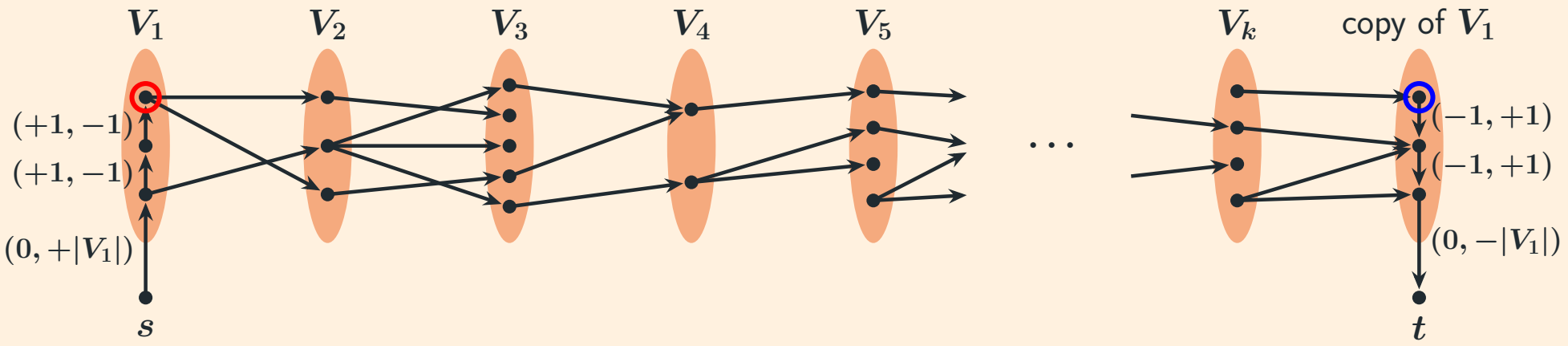

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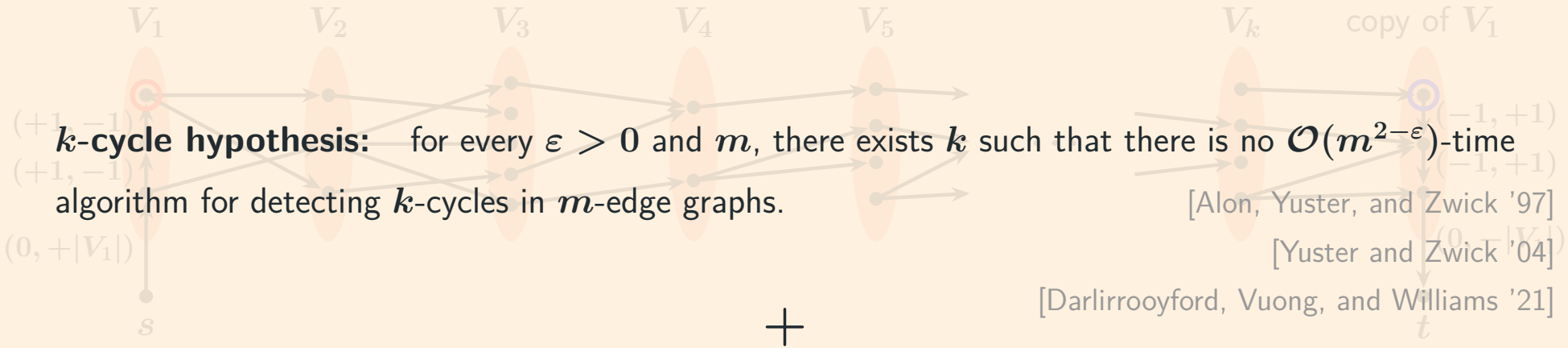

k-cycle hypothesis: for every $\varepsilon > 0$ and m, there exists k such that there is no $\mathcal{O}(m^{2-\varepsilon})$-time algorithm for detecting k-cycles in m-edge graphs.

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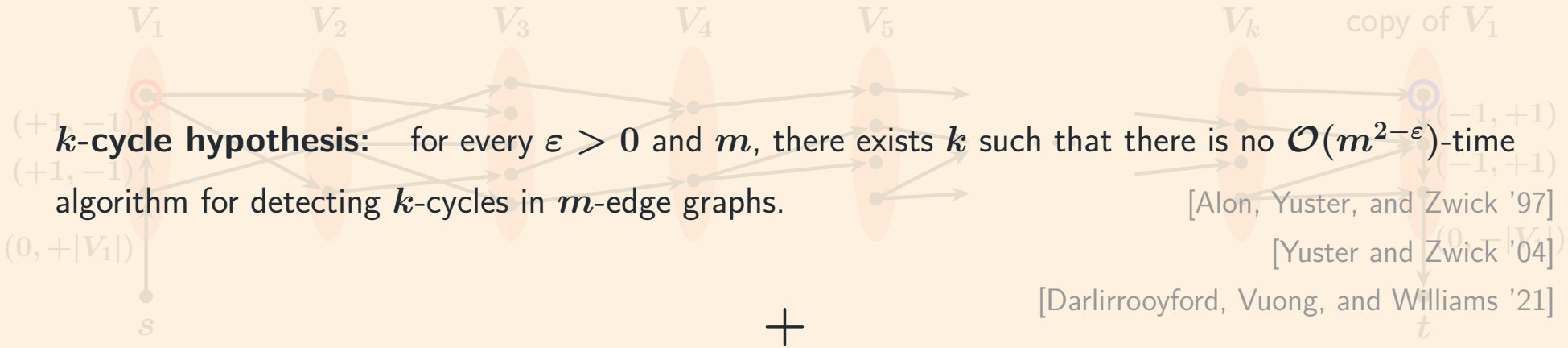

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$\Downarrow$

Theorem. Assuming the k-cycle hypothesis, there does not exist an $\varepsilon > 0$ and an $\mathcal{O}(n^{2-\varepsilon})$-time algorithm that decides coverability in 2-VASS. [Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Preview of this talk

Part 1: d-VASS

Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

Coverability in d-VASS conditionally requires $n^{2^{\Omega(d)}}$ time.

Parameterised reduction from k-clique detection.

Part 2: 2-VASS

Coverability in 2-VASS can be decided in n^C time.

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Linear-time reduction from k-cycle detection.

Part 3: 1-VASS

Coverability in 1-VASS can be decided in n^2 time

Coverability in (binary) 1-VASS is in AC^1.

Open Problem

What is the *exact* complexity of of coverability in 1-VASS ?

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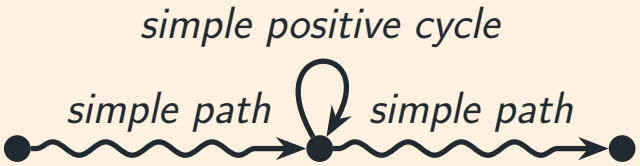
or 

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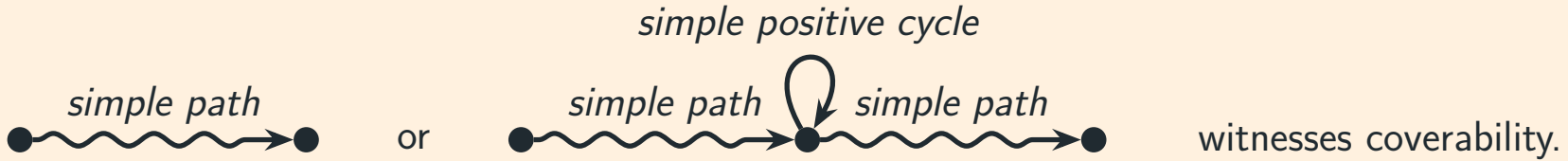
$\Longrightarrow$ 

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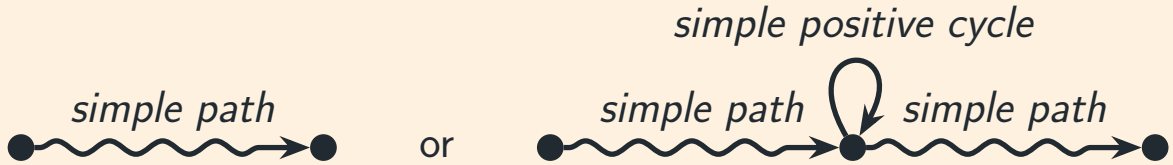
$\Longrightarrow$ 

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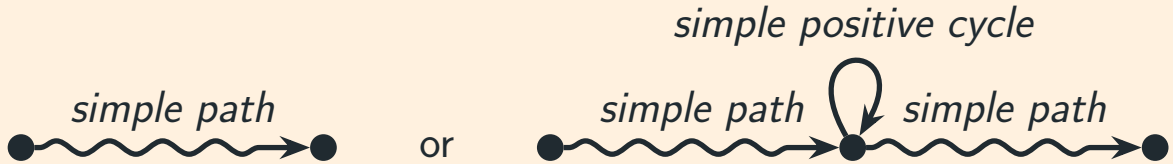
$\Longrightarrow$ 

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This matches the state-of-the-art for shortest paths in $\mathbb{N}$-weighted directed graphs. [Cook '85]

An Efficient Algorithm for Coverability in 1-VASS

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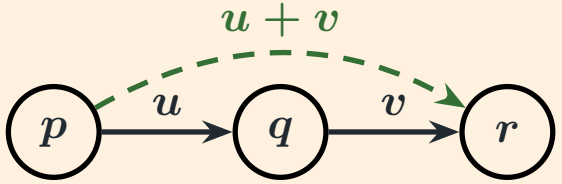

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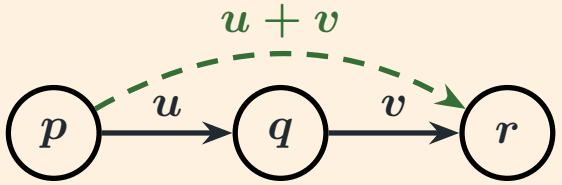

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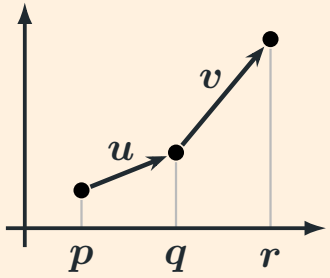

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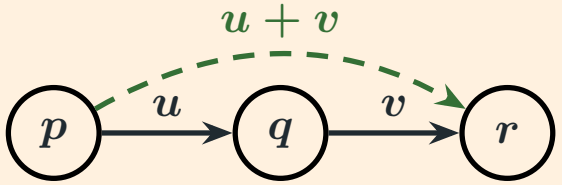

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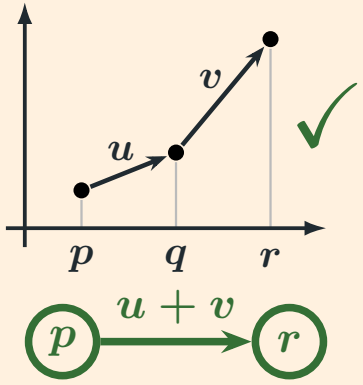

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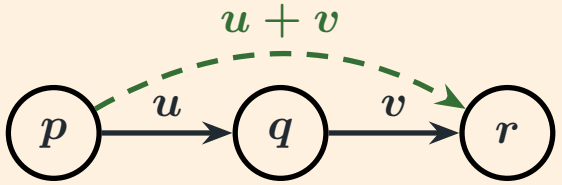

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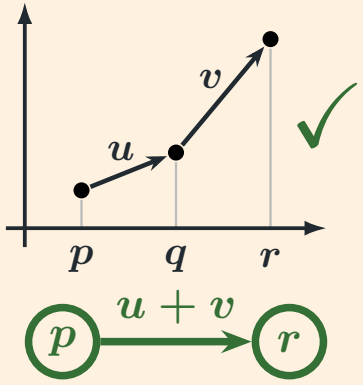

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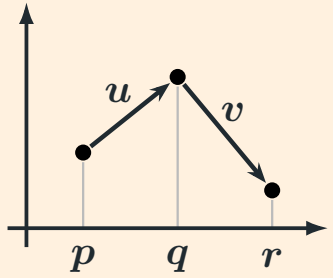

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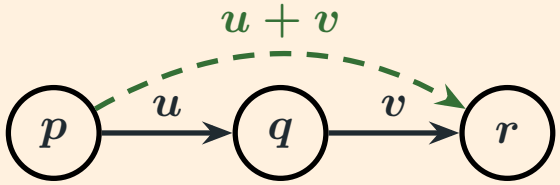

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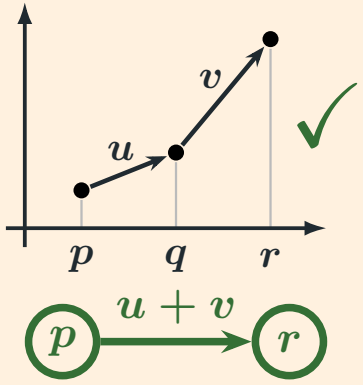

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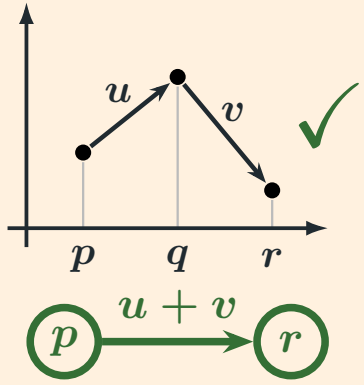

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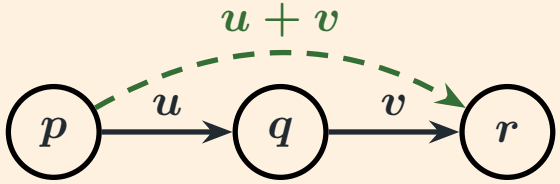

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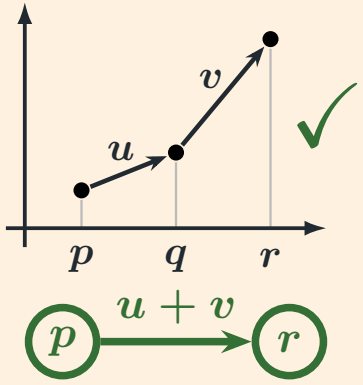

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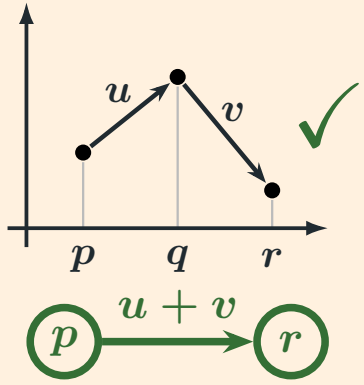

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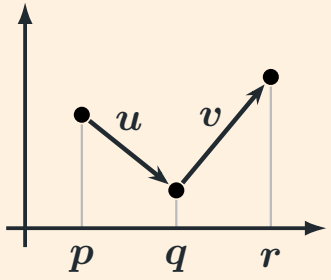

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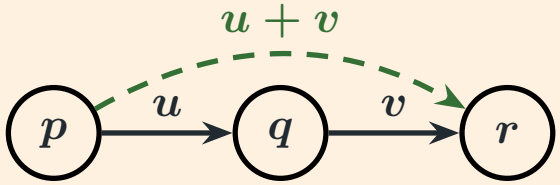

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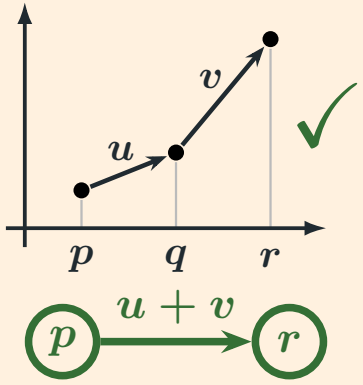

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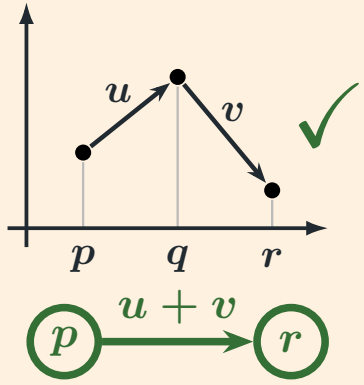

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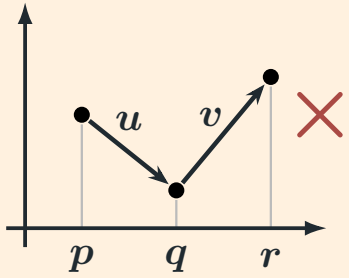

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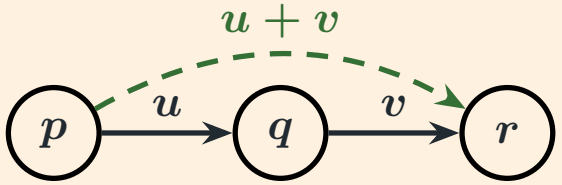

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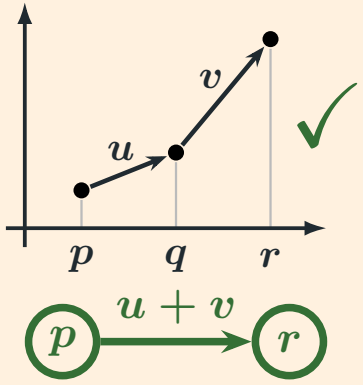

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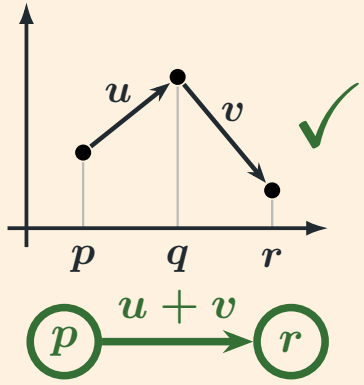

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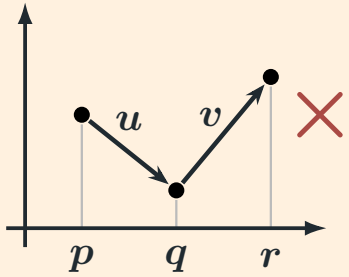

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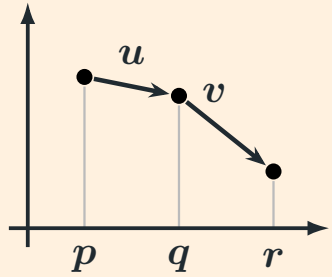

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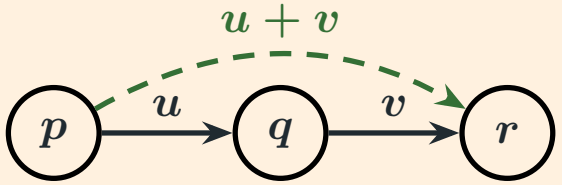

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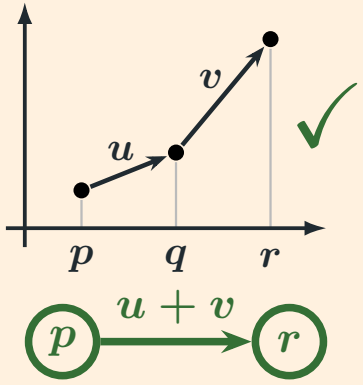

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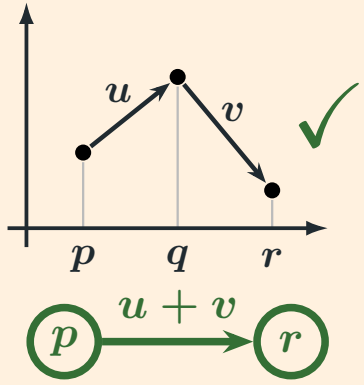

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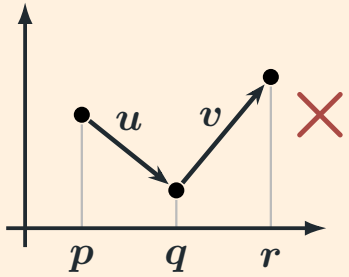

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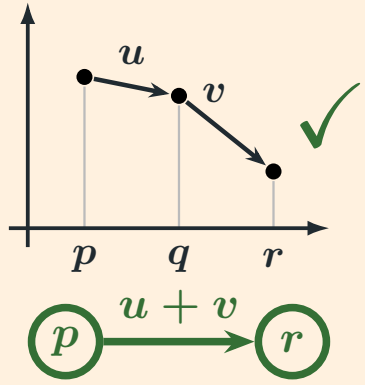

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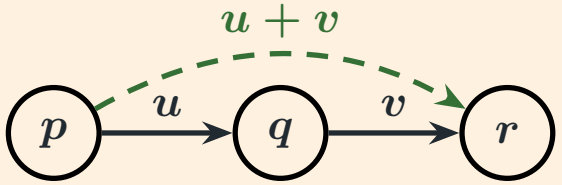

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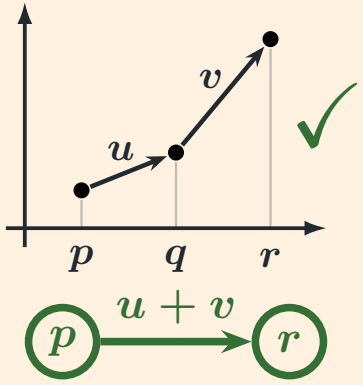

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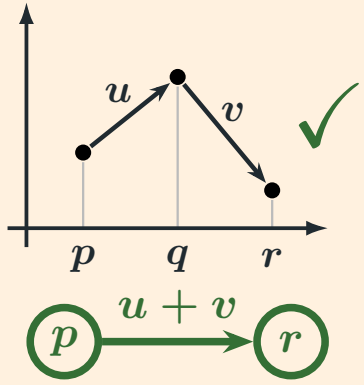

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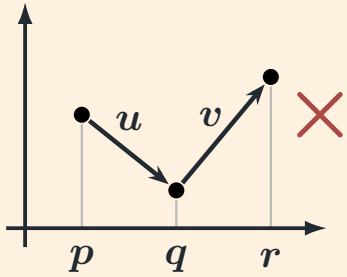
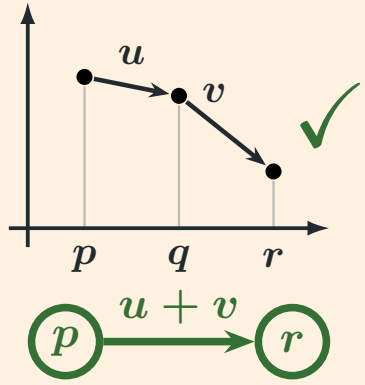


Do not add shortcut transition

Scenario 4

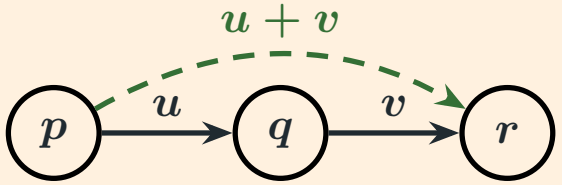


STEP i: Repeat $k = 2\lceil \log n \rceil$ many times.

An Efficient Algorithm for Coverability in 1-VASS

STEP 0: Let $\mathcal{V}_0$ be the given 1-VASS.

STEP 1: Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":



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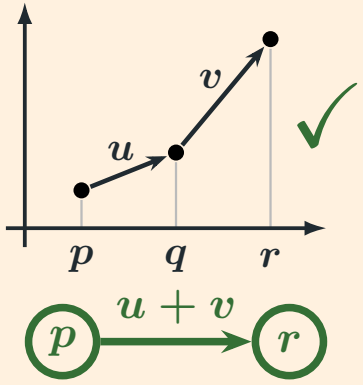
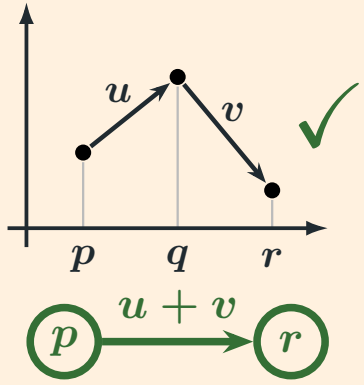
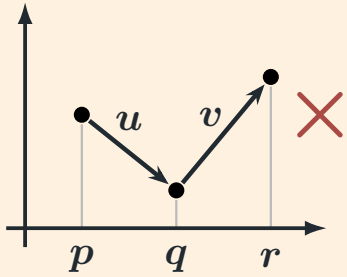


Scenario 2



Scenario 3



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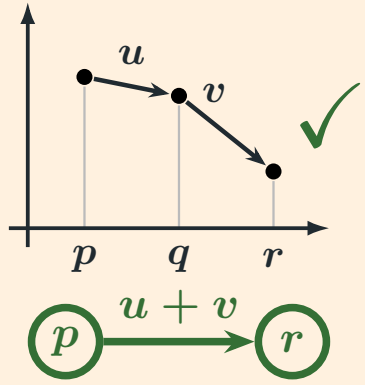

STEP i: Repeat $k = 2\lceil \log n \rceil$ many times.

Claim. There is a covering run in $\mathcal{V}_0$ if and only if there is a covering run of **length** ≤ 2 in $\mathcal{V}_k$.

What is the Fine-Grained Complexity of Coverability in 1-VASS?

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Theorem. Coverability in 1-VASS can be decided in $\mathcal{O}(n^2)$ time.

Observation. One need not take non-positive weight cycles.

Idea. Modify the $\mathcal{O}(|V||E|)$Bellman-Ford algorithm for shortest paths in integer-weighted graphs.

Theorem. Coverability in (binary-encoded) 1-VASS is in AC^1. [Shakiba, S-B, and Zetzsche '25]

This matches the state-of-the-art for shortest paths in $\mathbb{N}$-weighted directed graphs. [Cook '85]

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Theorem. There is a *randomised* algorithm which computes a shortest path tree from s or finds a negative cycle in n polylog(n) time. [Bernstein, Nanongkai, and Wulff-Nilsen '22] [Bringmann, Cassis, and Fischer '23]

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Precise Complexity Analysis of Coverability in Vector Addition Systems

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Theorem. Coverability in 1-VASS can be decided in $\mathcal{O}(n^2)$ time and is in AC^1.

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Thank You!

Presented by Henry Sinclair-Banks, MPI-SWS, Kaiserslautern, Germany

INFINITY 2026, Royal Holloway, University of London, Egham, UK

Presentation made with BeamerikZ