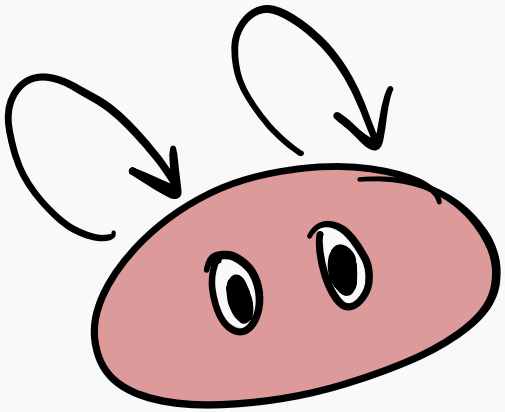
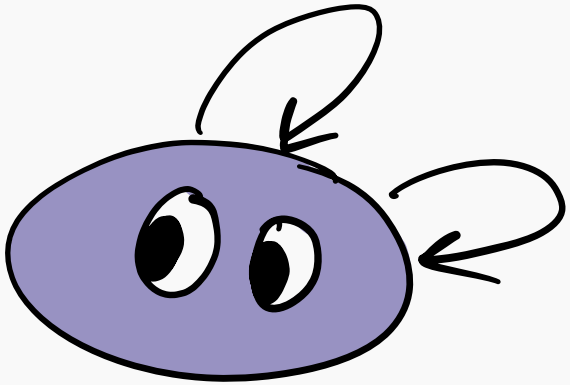
Intersecting Dense Automata

Constructions, Complexity and Certificates



Neha Rino

University of Warwick



Joint work with Dmitry Chistikov

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Centre for Discrete Mathematics and its Applications (DIMAP)
Feuer International Scholarship in AI
Engineering & Physical Sciences Research Council, UK

Plan

I NFA K-INTERSECTION (EMPTINESS)

II PRODUCT CONSTRUCTIONS

III REDUCTION FROM K-CLIQUE

IV CERTIFICATES AND SETH

Problems of Interest

Input: k NFAs $\mathcal{A}_1, \ldots \mathcal{A}_k$.

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Intersection

An NFA for $\bigcap_{i=1}^{k} \mathcal{L}(\mathcal{A}_i)$.

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Intersection Emptiness

Is $\bigcap_{i=1}^{k} \mathcal{L}(\mathcal{A}_i) = \emptyset$?

Problems of Interest

Input: k NFAs $\mathcal{A}_1, \ldots \mathcal{A}_k$.

Intersection

An NFA for $\bigcap_{i=1}^{k} \mathcal{L}(\mathcal{A}_i)$.

NFA k-IE

Is $\bigcap_{i=1}^{k} \mathcal{L}(\mathcal{A}_i) = \emptyset$?

Complexity of Intersecting NFAs $\mathcal{A}_1, \ldots, \mathcal{A}_k$

Solution: Construct the direct product of $\mathcal{A}_1, \ldots, \mathcal{A}_k$.

[Rabin and Scott, 1959]

Intersecting NFAs $\mathcal{A}_1, \ldots, \mathcal{A}_k$ with n states and m transitions

Solution: Construct the direct product of $\mathcal{A}_1, \ldots, \mathcal{A}_k$.

Intersection

Size of the direct product is $O(m^k) = O(n^{2k})$.

[e.g. Esparza and Blondin, Automata Theory: An Algorithmic Approach, 2023]

NFA k-IE

DFS running time = $O(m^k) = O(n^{2k})$.

[e.g. Baier and Katoen, Principles of Model Checking, 2008]

Intersecting NFAs $\mathcal{A}_1, \ldots, \mathcal{A}_k$ with n states and m transitions

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$\bigcap_{i=1}^{k} \mathcal{L}(\mathcal{A}_i)$ can require $\Omega(n^k)$ states

[Maslov, 1970].

NFA k-IE

DFS running time = $O(m^k) = O(n^{2k})$.

Faster than $\Omega(n^k)$-time $\implies$ breakthrough in solving SAT

[Wehar, 2014].

Today's question

Intersection and intersection emptiness (NFA k-IE)

Can we close this gap between $O(n^{2k})$ and $\Omega(n^k)$?

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Our contribution

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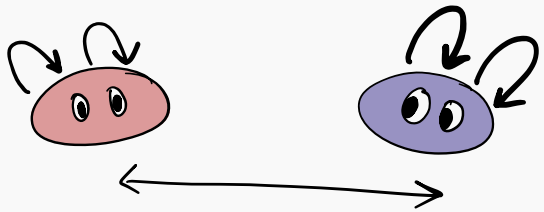
- An $O(mn^{k-1}) = O(n^{k+1})$-sized NFA for the intersection.
- A conditionally optimal algorithm for NFA k-IE.
- A faster certificate scheme for NFA k-IE.

Plan

I. NFA k-INTERSECTION (EMPTINESS)

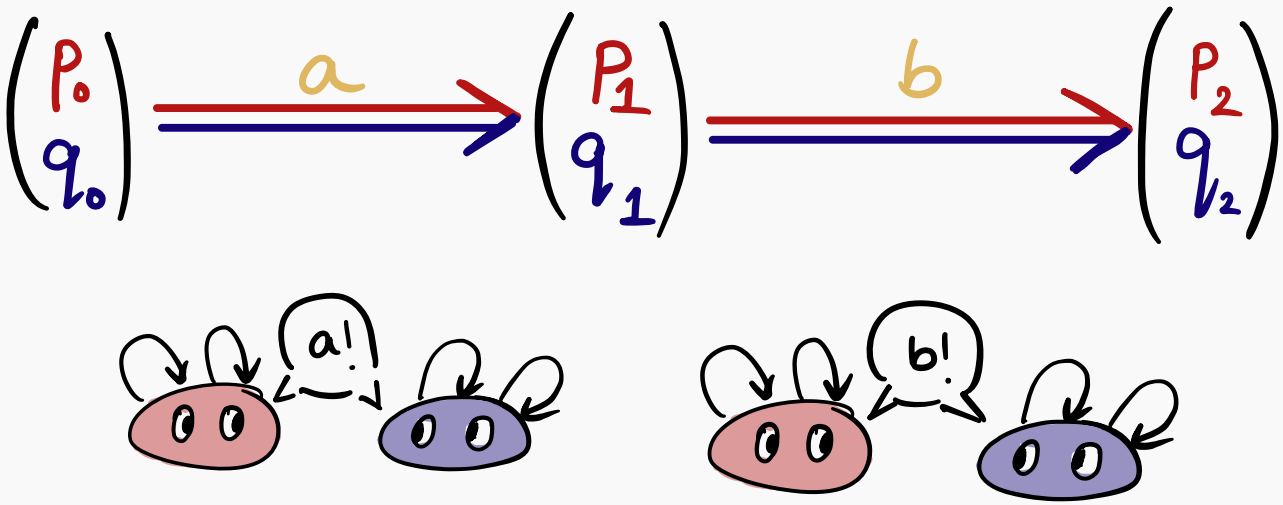


II PRODUCT CONSTRUCTIONS

III REDUCTION FROM k-CLIQUE

IV CERTIFICATES AND SETH

I. Direct product



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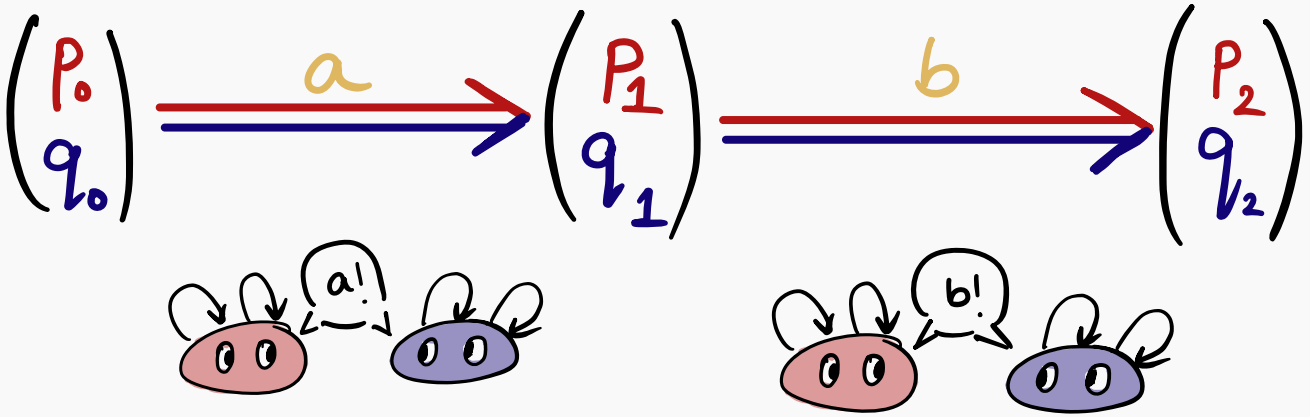
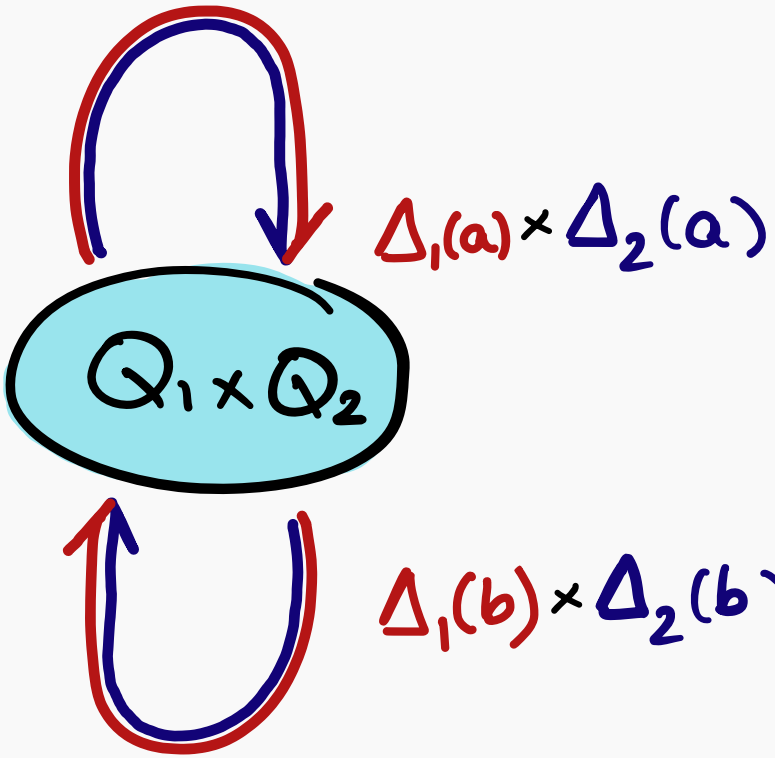


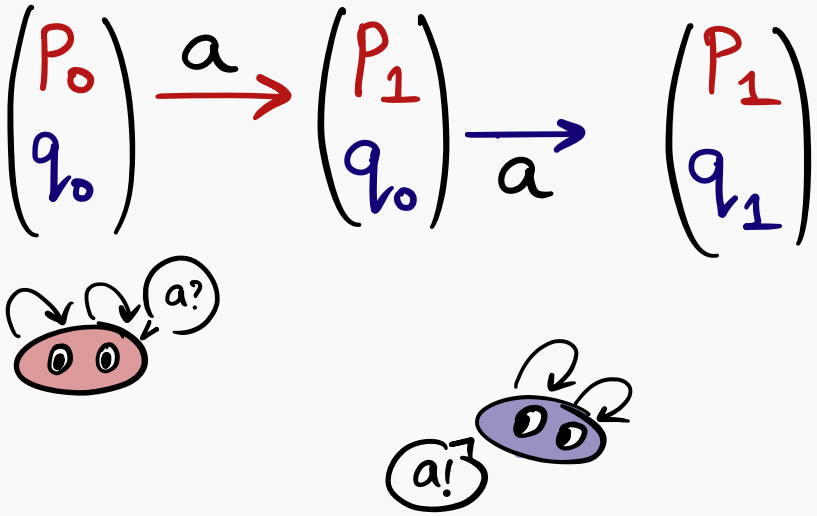


$$
\begin{aligned}
& |\Delta_1(a)| \times |\Delta_2(a)| \\
+\ & |\Delta_1(b)| \times |\Delta_2(b)| \\
=\ & O(m^2)
\end{aligned}
$$

II. Idea: Silently Nodding

Central Idea

Interleave transitions?



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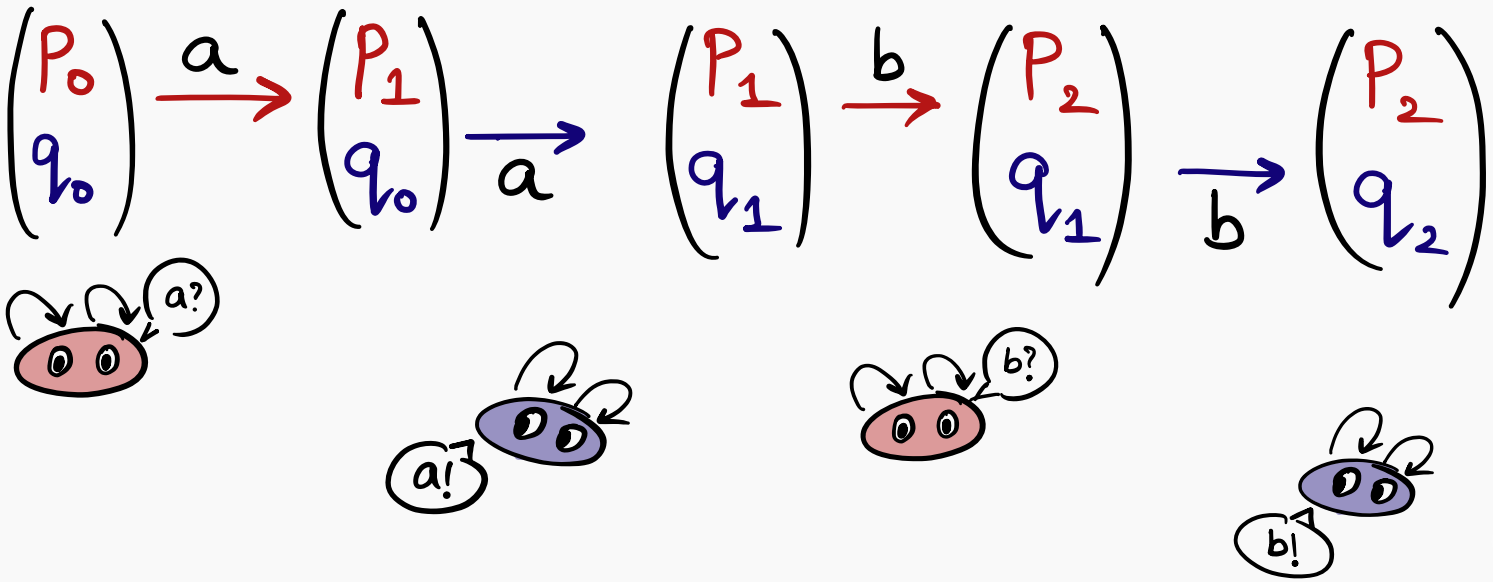
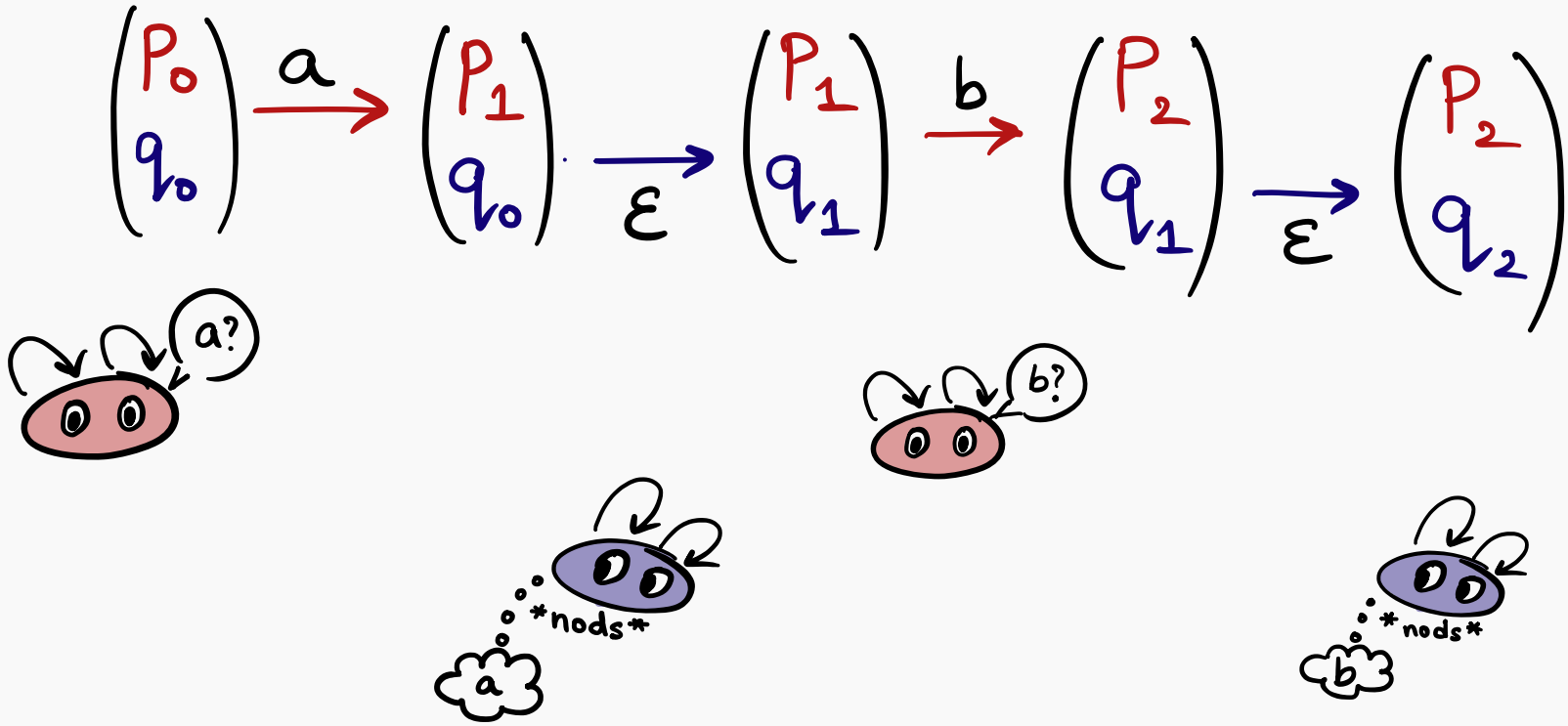


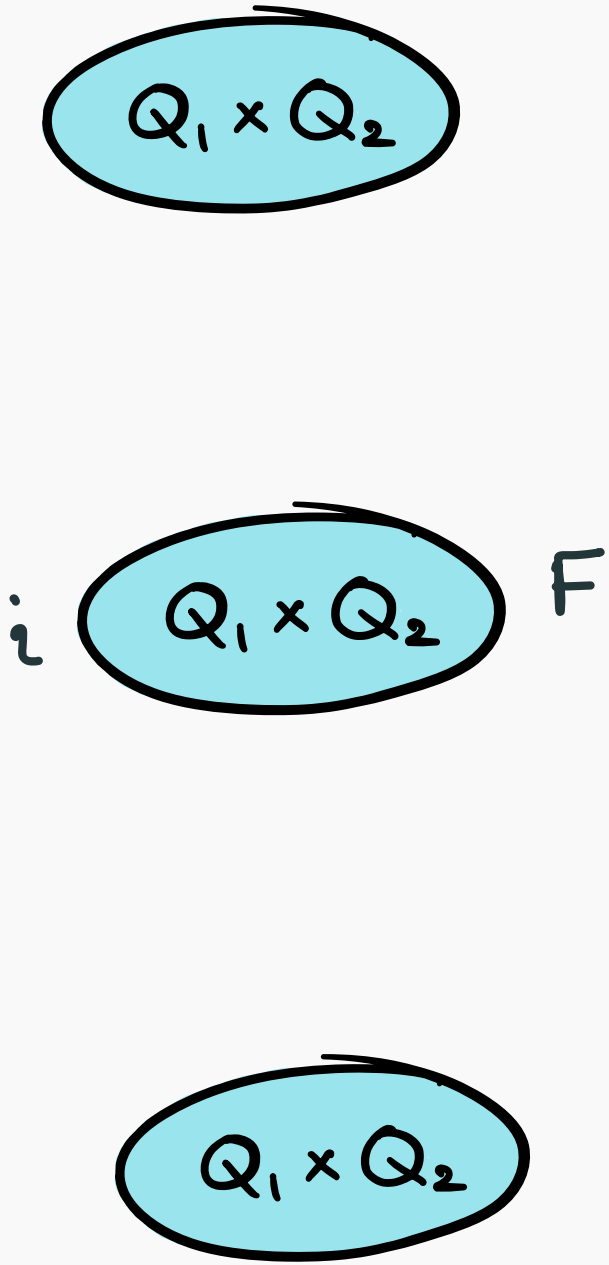
II. Idea: Silently Nodding

Central Idea

Interleave transitions while preserving the word read.



II. Nodding Product



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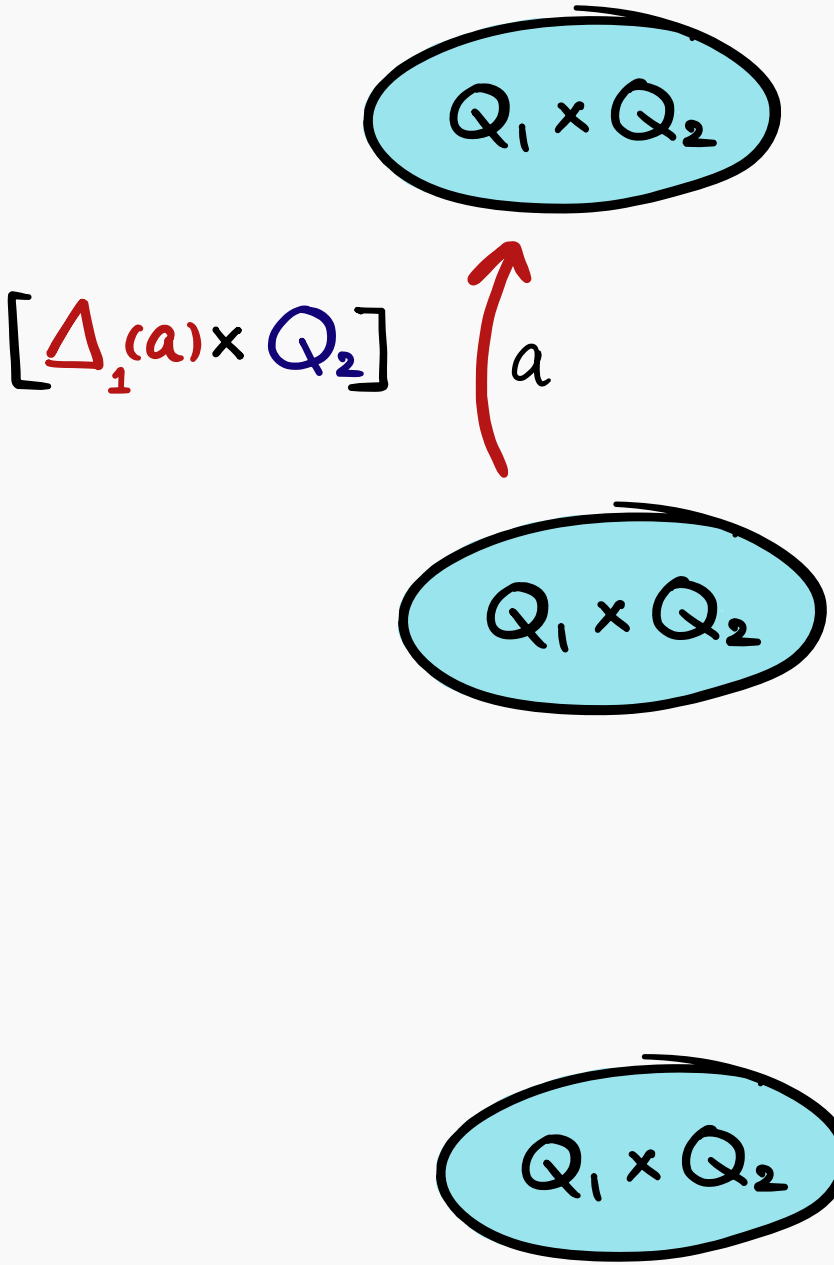

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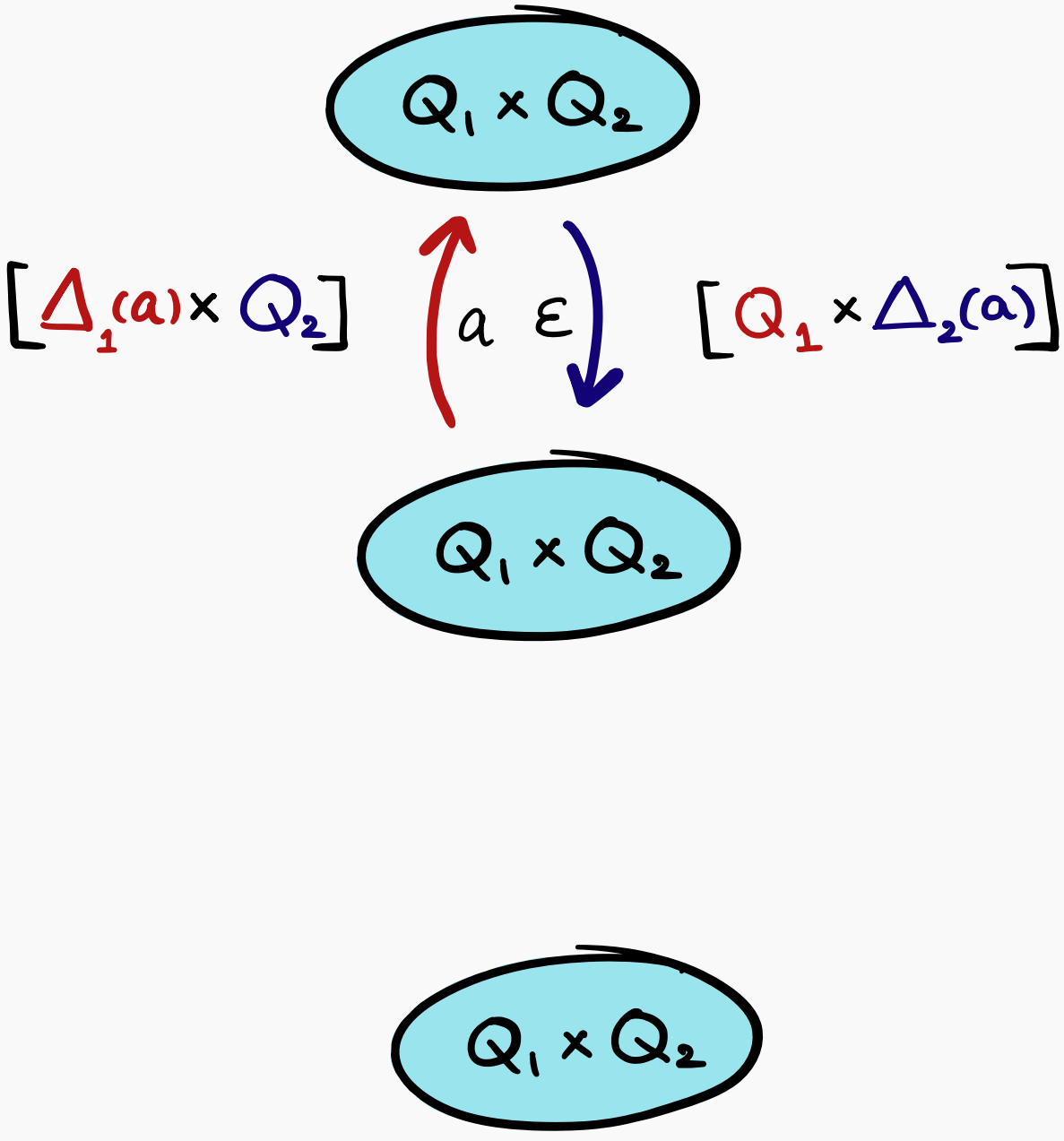

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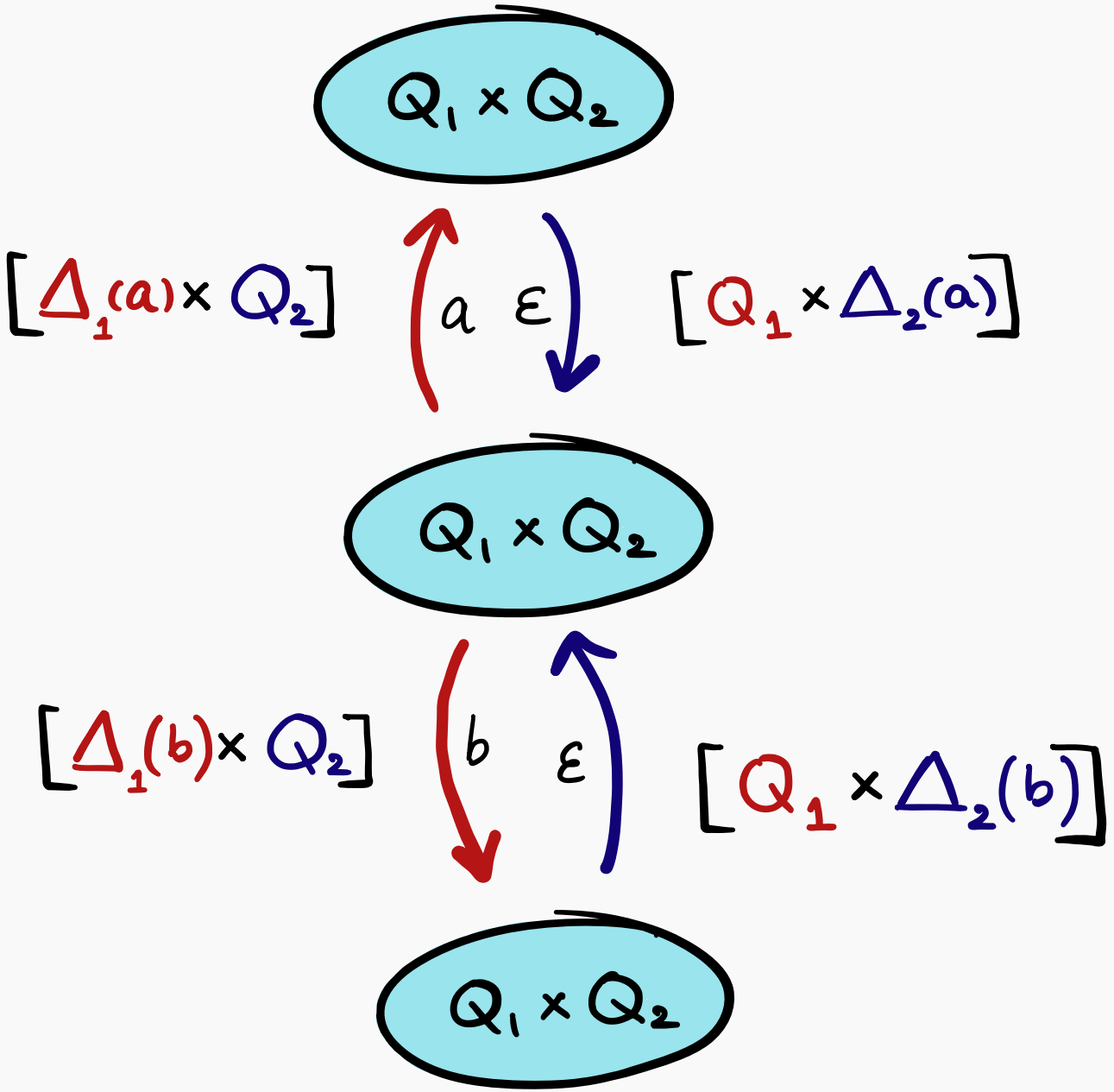
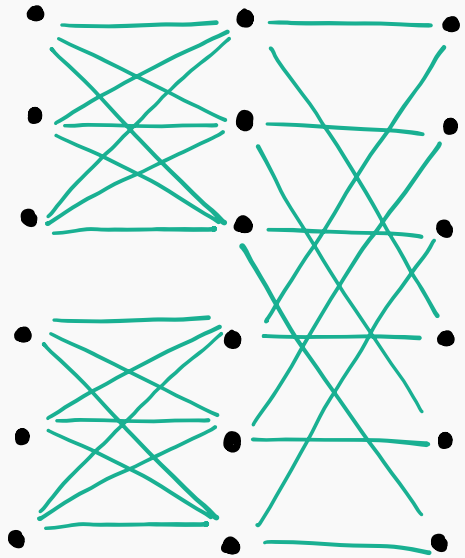


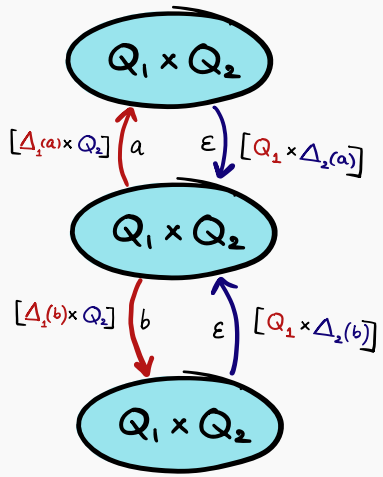
II. Nodding Product

Direct Product
(Dense)



Nodding product
(Sparse)

II. Size of the nodding product



The nodding product recognises the intersection, using

- $O(n^2)$ states
- $O(mn)$ transitions

II. Size of the nodding product

Theorem (1)

Given k NFAs over an alphabet of size ℓ, the nodding product recognises their intersection using

- $O(k\ell \cdot n^k)$ *states*
- $O(k \cdot mn^{k-1})$ *transitions.*

Plan

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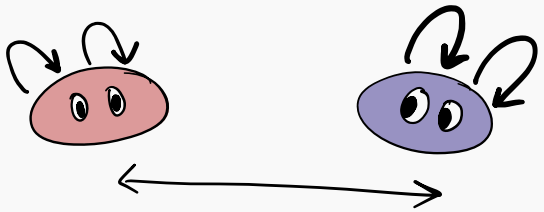

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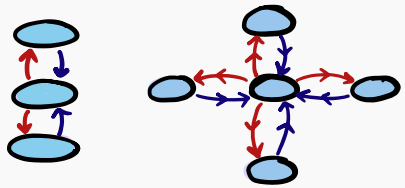

III REDUCTION FROM K-CLIQUE

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I. Algorithm for NFA k-IE

Algorithm: Search in the nodding product, constructed on the fly. Check if some final state is reachable.

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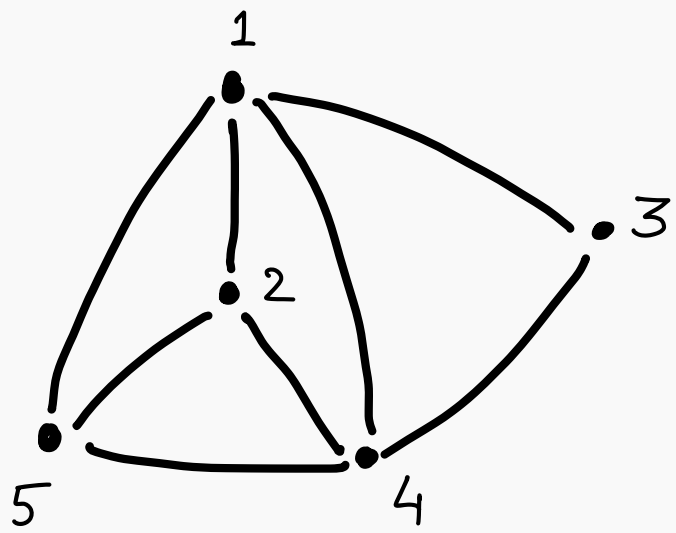
Algorithm: Search in the nodding product, constructed on the fly. Check if some final state is reachable.

Running time: The number of reachable states and transitions in the nodding product is $O(k \cdot mn^{k-1})$.

II. Spot the pattern

Pop quiz

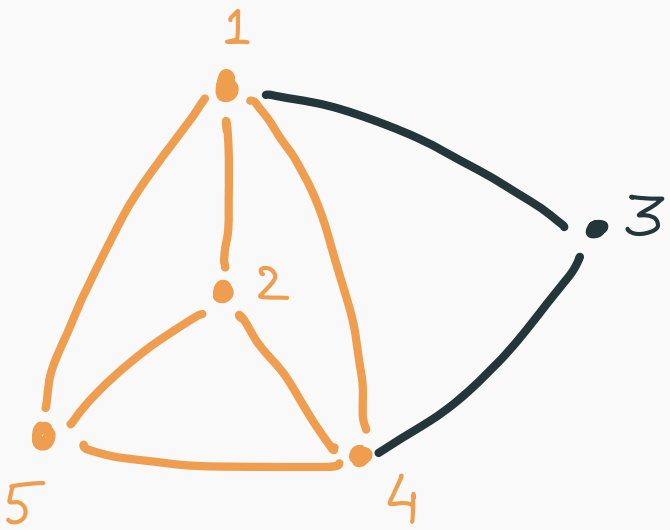
Does the following graph contain 4 vertices that are all pairwise adjacent?



II. Spot the pattern

k-Clique problem

Given a graph G, does it contain k vertices which are all pairwise adjacent?



Yes! {1, 2, 4, 5}

II. The k-Clique problem

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Given a graph G, does it contain k vertices which are all pairwise adjacent?

The best known running times (upto polylog factors) for k-Clique algorithms are:

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The best known running times (upto polylog factors) for k-Clique algorithms are:

$O(n^{\omega\lceil\frac{k}{3}\rceil})$ time using fast matrix multiplication.

[Nešetřil and Poljak, 1985]

$O(n^k)$ time without fast matrix multiplication.

[Abboud et al, 2024]

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$O(n^k)$ time without fast matrix multiplication.

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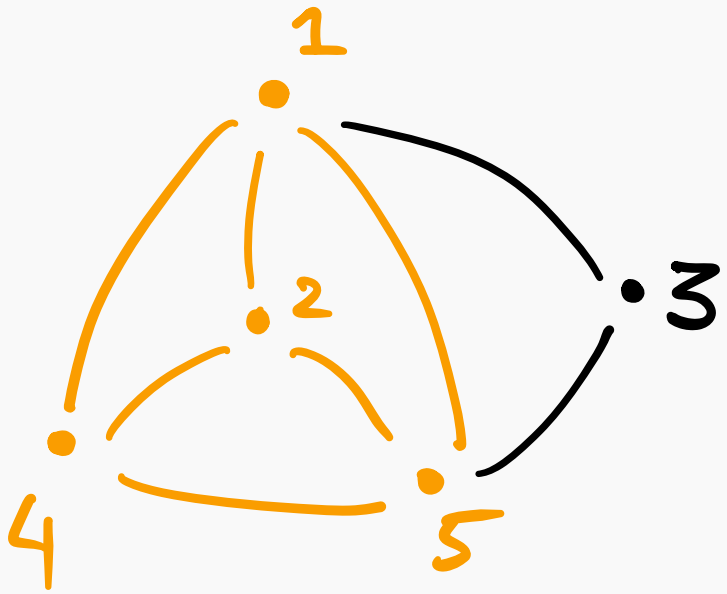
$n \times n$ matrix multiplication is in $O(n^\omega)$ time, where $\omega < 2.372$.

II. Finding a 4-clique using 3 languages

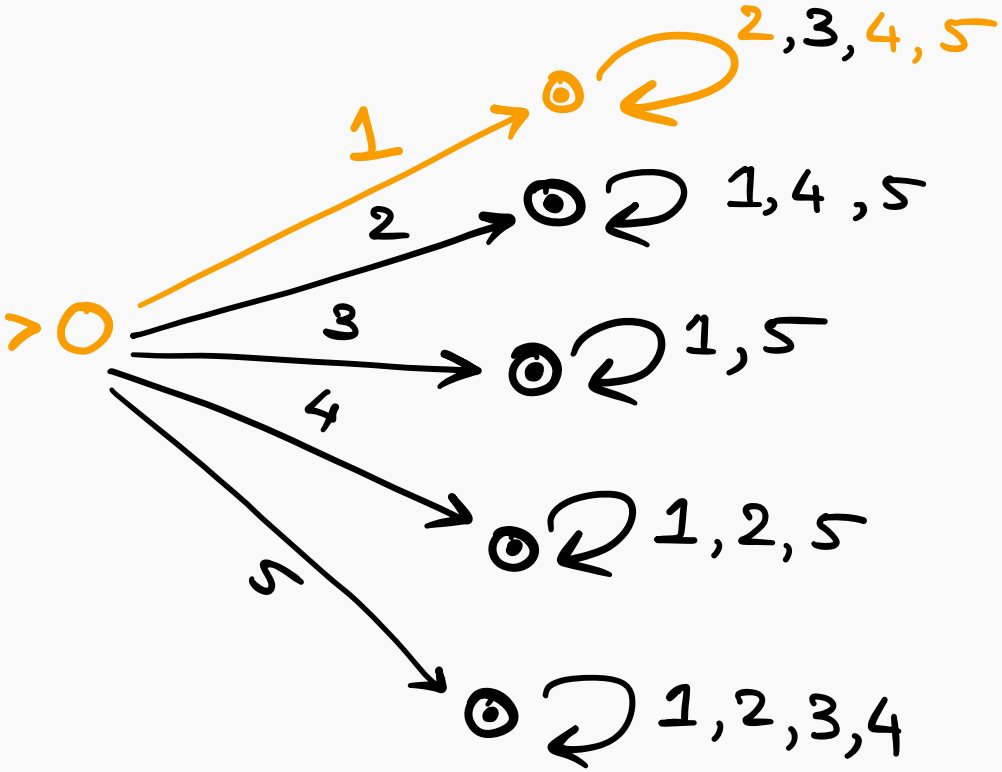
There is a 4 clique in G if:

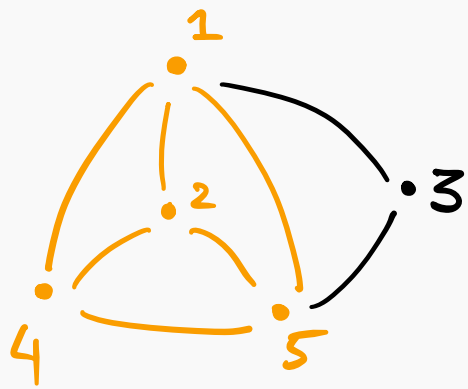
1. $\exists\ u_1, u_2, u_3, u_4$
2. u_1 is adjacent to u_2, u_3, u_4
3. u_2 is adjacent to u_3, u_4
4. u_3 is adjacent to u_4



II. The three NFAs

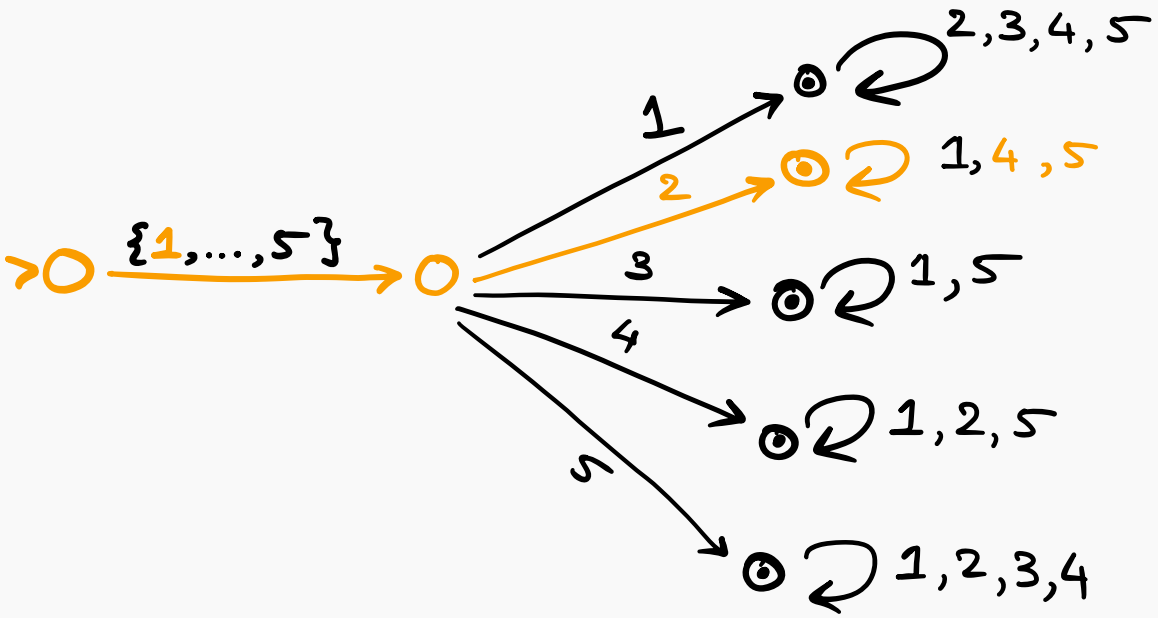
$$L(\mathcal{A}_1) = \bigcup_{u \in V} u \cdot (\mathrm{Neigh}(u))^*$$

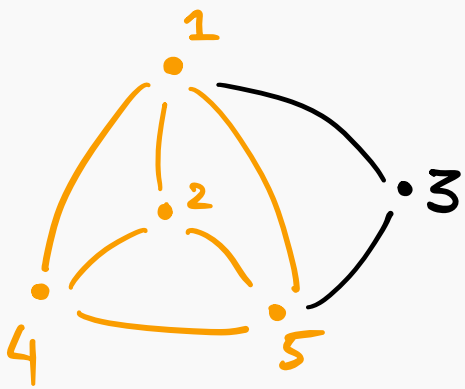




II. The three NFAs

$$L(\mathcal{A}_2) = V \cdot \left(\bigcup_{u \in V} u \cdot (\text{Neigh}(u))^* \right)$$





III. Conditional Optimality

Theorem (3)

If NFA k-IE has an $O((mn^{k-1})^{1-\epsilon})$ time algorithm, then $(k+1)$-Clique has an $O((n^{k+1})^{1-\epsilon})$ time algorithm.

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If NFA k-IE can be sped up without fast matrix multiplication, then so can k-Clique.

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If NFA k-IE can be sped up without fast matrix multiplication, then so can k-Clique.

Combinatorial k-Clique Hypothesis [Abboud, Backurs, Williams, 2018]

There is no $O((n^{k+1})^{1-\epsilon})$ time algorithm for $(k+1)$-Clique that avoids fast matrix multiplication, regardless of $\epsilon > 0$.

III. Conditional Optimality

Corollary (4)

If NFA k-IE has an $O((mn^{k-1})^{1-\epsilon})$ time algorithm that avoids fast matrix multiplication, the Combinatorial k-Clique Hypothesis is false.

- It was known that triangle detection reduces to NFA 2-IE [Potechin and Shallit, 2020], even to the special case of NFA Acceptance.

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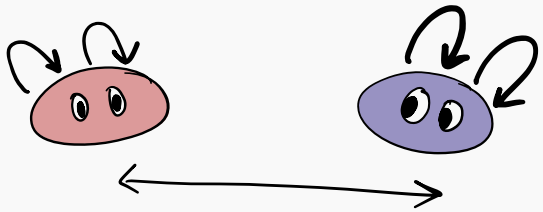
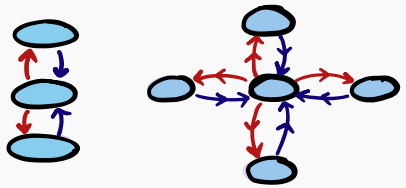
- [Bringmann et al, 2024] conjecture that the NFA Acceptance problem, given an m transition NFA and a word w, already needs $m|w|$ time.

Plan

I NFA K-Intersection (Emptiness)



II Product Constructions



III Reduction From K-Clique

IV Certificates and Seth

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We now have a conditionally optimal algorithm for NFA k-IE!

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We now have a conditionally optimal algorithm for NFA k-IE!

Let's all implement it in (LASH, SPIN, Walnut, MATA, PathFinder) because it uses NFA intersection as a subprocedure.

But what if our implementation had a bug? How would we even notice?

I. A Certifying Algorithm

Certifying Algorithm (e.g. [McConnell et al 2010])

A certifying algorithm doesn't just output yes or no, it also hands you an independently verifiable certificate of correctness.

II. Certifying intersection non-emptiness (Warmup)

Input k NFAs $\mathcal{A}_1, \ldots, \mathcal{A}_k$

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A certificate A word in $L(\mathcal{A}_1) \cap \cdots \cap L(\mathcal{A}_k)$

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A very convenient certificate k accepting runs, one in each NFA, over the same short word.

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A very convenient certificate k accepting runs, one in each NFA, over the same short word.

Verifier Verifies the runs are short, accepting, and all read the same word in $O(n^k)$ time.

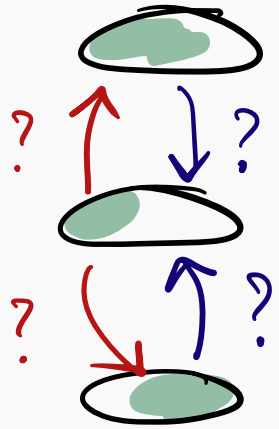
III. Certifying intersection emptiness

If $k = 1$, this is just non-reachability in a graph.

Input An NFA $\mathcal{A}$

Certificate A cut in the transition graph

Verifying a cut in the nodding product automaton involves enumerating all of its transitions.



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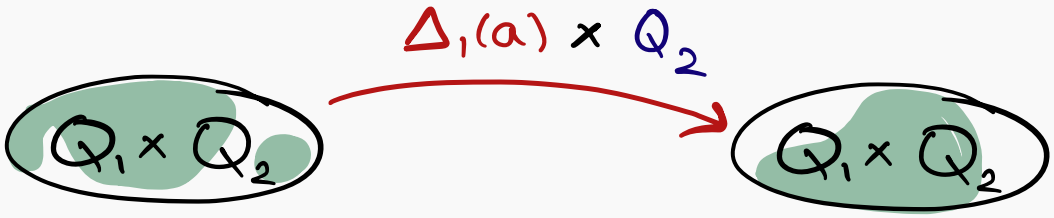
Verifying a cut in the nodding product automaton involves enumerating all of its transitions.

Hence, this takes $O(k \cdot mn^{k-1})$ time, same as solving NFA k-IE from scratch.

III. Idea: The shape of the nodding product

Transitions in the nodding product are grouped into volleys.



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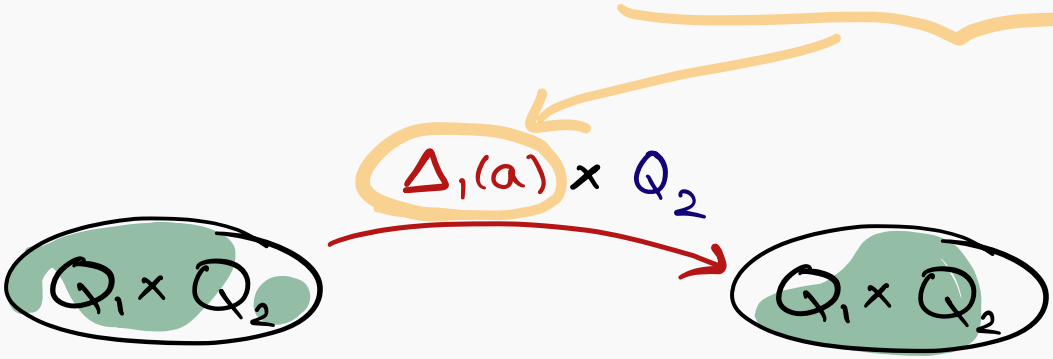
Transitions in the nodding product are grouped into volleys.

Each volley updates only one given component, by one letter's transitions.



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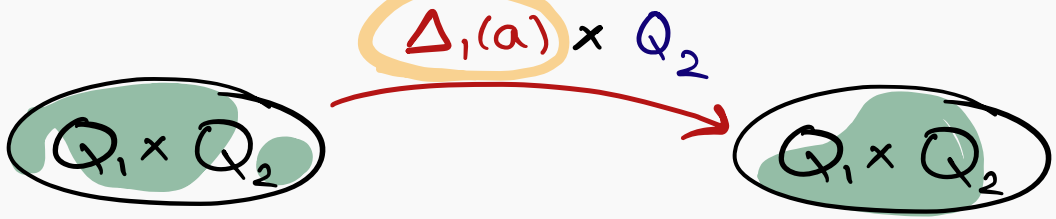
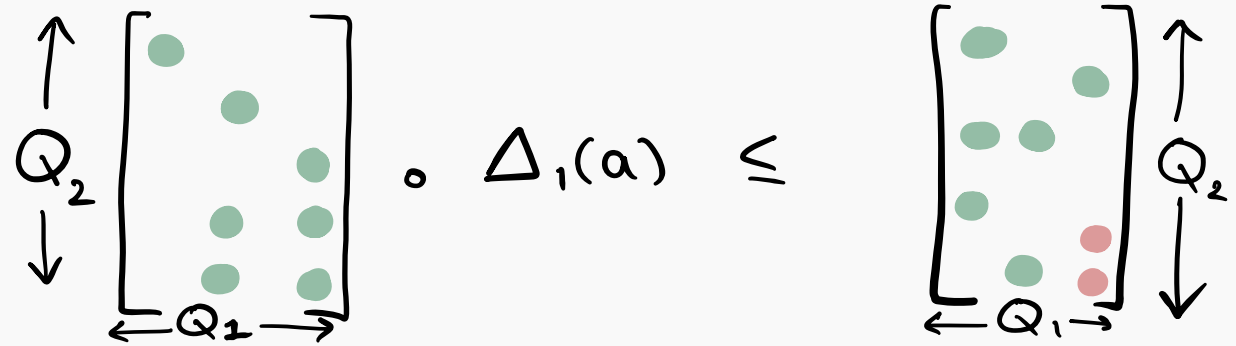


Using matrices, verify that a volley does not leave the cut using one $n^{k-1} \times n$ by $n \times n$ matrix multiplication.



III. Certifying intersection emptiness

Theorem (5)

NFA k-IE has a certificate system that can be verified in time $O(k\ell \cdot n^{k+\omega-2})$.

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NFA k-IE has a certificate system that can be verified in time $O(k\ell \cdot n^{k+\omega-2})$.

For sufficiently dense automata, this is faster than solving

NFA k-IE!

IV. SAT-based lower bounds

- P $\neq$ NP hypothesis: There is no $O(n^k)$-time algorithm for solving SAT

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- P $\neq$ NP hypothesis: There is no $O(n^k)$-time algorithm for solving SAT
- Strong Exponential Time Hypothesis (SETH): There is no deterministic $1.999..9^n$-time algorithm for solving SAT
- Nondeterministic Strong Exponential Time Hypothesis (NSETH): There is no nondeterministic $1.999..9^n$-time algorithm for solving TAUT

IV. SETH for NFA k-IE

Theorem (Wehar, 2014)

An $O(n^{k-\epsilon})$ time algorithm for DFA k-IE would refute SETH.

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Theorem (Carmosino et. al., 2016)

Whenever a problem has a certificate system that runs in time strictly below the best known upper bound, a tight SETH-based lower bound would contradict NSETH.

IV. A tight SETH lower bound is unlikely

Corollary (6)

An (mn^{k-1})-lower bound for NFA k-IE from SETH would contradict NSETH.

- Since there is an $\Omega(n^k)$ SETH lower bound [Wehar, 2014], a certificate system faster than $O(n^k)$ would also refute NSETH.

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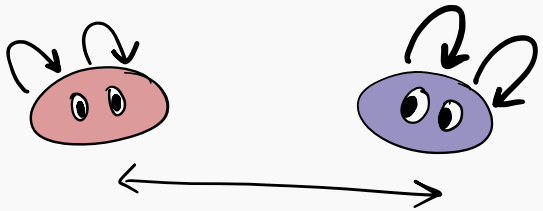
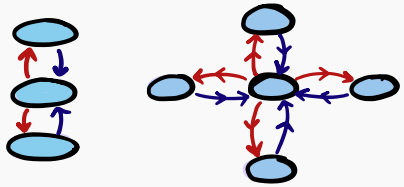
- Certifying language emptiness/non-acceptance quickly using fast matrix multiplication is also used in [Bringmann et al, 2024] and [Chistikov et al, 2022].

II. Summary

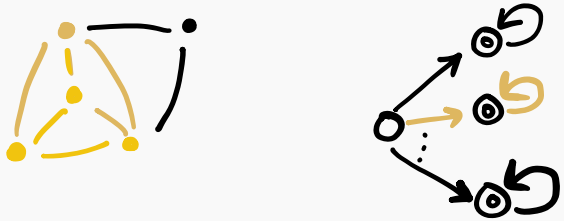
NFA K-Intersection (Emptiness)



Product Constructions



Reduction From k-Clique



Certificates and SETH