Fine-Grained Complexity and Formal Languages

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Classical Complexity Versus Fine-Grained Complexity

Classical computational hardness

Hardness hypothesis
SAT requires super-polynomial time.
Rather independent of machine model.

Polynomial-time reduction
Purpose: If Q can be solved in poly-time, then SAT can (and any problem in NP).

Conditional hardness
Conclusion: Q requires super-polynomial time (on any machine model).

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- 3-SAT can be solved in time $\mathcal{O}\left(2^{\text{polylog}(n+m)\cdot\sqrt{n+m}}\right)$?

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Motivation

(S)ETH bounds can be practically more relevant than pure NP-hardness assertions.
The running-time bounds are more concrete than just being non-polynomial.

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Theorem (Williams 2005 / Williams, Yu 2015)
SETH implies OVH.

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Reductions: Classical Versus Fine-Grained

Classical Setting

- Mostly many-one reductions between decision problems,
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Basic Facts on Fine-Grained Reductions

- If problem A is (a,b)-reducible to B and if B can be solved in time $\mathcal{O}\left(b(n)^{1-\varepsilon}\right)$, then A can be solved in time $\mathcal{O}\left(a(n)^{1-\delta}\right)$.
- If A is (a,b)-reducible to B and B is (b,c)-reducible to C, then A is (a,c)-reducible to C.

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NFA	unary	binary
Non-Universality	NP-complete	PSPACE-complete
Inequivalence	NP-complete	PSPACE-complete
Intersection Non-Emptiness	NP-complete	PSPACE-complete

$\rightsquigarrow$ Focus on unary (tally) languages.

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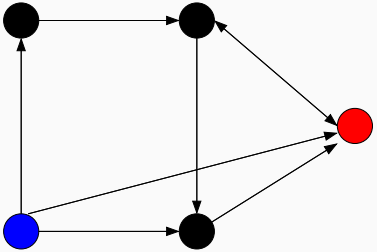
UNIVERSALITY hence models the folllowing scenary:

Can somebody living in the blue circle visit the red one in k steps, for **any** number k ?



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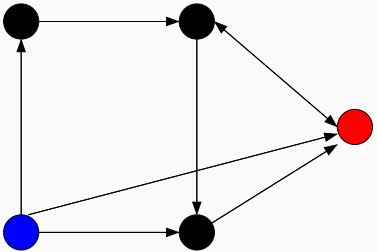

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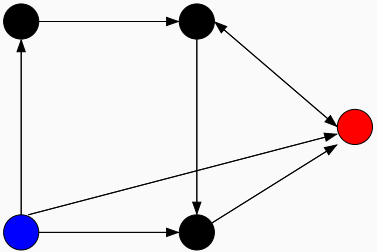

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So, possibly those mostly interested in graphs may bear with us ...

Non-Universality for Tally NFAs: NP-hardness by Stockmeyer & Meyer 1973

Reduction from 3-SAT ☺ (n variables, m clauses)

Non-Universality for Tally NFAs: NP-hardness by Stockmeyer & Meyer 1973

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Idea: Codify assignments by Chinese remainder.

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Recall: $p_n \sim n \ln n$.

$\exists$ NFA A_0 for $L_0 := \bigcup_{k=1}^{n} \bigcup_{j=2}^{p_k - 1} \{a\}^j \{a^{p_k}\}^*$ with $\leq np_n \sim n^2 \ln n$ many states.

L_0 collects words that do not encode assignments, i.e., syntactic trash. Ex.: $aa \in L_0$.

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$i_j(\ell)$ is the index of the ℓ^{th} variable in clause c_j.

As $p_{i_j(1)} \cdots p_{i_j(|c_j|)} \leq p_n^3$ (3-SAT), L_j is accepted by a DFA with $\leq p_n^3 \leq n^3 \cdot (\ln n)^3$ many states.

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Observe: $L_0 \cup L_1 \cup \cdots \cup L_m = \{a\}^*$ iff given 3-SAT formula is unsatisfiable.

Altogether, the NFA has $\leq mp_n^3 \approx n^4 (\log n)^3$ states (after sparcification): quite a many! ☹

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Cor.: Unless ETH fails, for any $\epsilon > 0$, there is no $\mathcal{O}^*(2^{o(q^{1/4-\epsilon})})$-time algorithm for deciding, given a tally NFA A on q states, whether $L(A) = \{a\}^*$.

What Are the “Correct Bounds”?

The algorithmic side:

Textbook algorithm (conversion into DFAs) yields $\mathcal{O}^*(2^q)$-time algorithm.

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Thm.: Unless ETH fails, there is no $\mathcal{O}^*(2^{o(m)})$-time algorithm for deciding if a given m-edge graph has a (proper) 3-coloring.

Standard proof ... / W.l.o.g., $m = O(n)$.

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Now, mimic proof of Stockmeyer / Meyer, reducing from 3-Coloring.

variables $\approx$ vertices / clauses $\approx$ edges

(a) Coloring condition has two vertices, not three variables $\rightsquigarrow$ improvement

(b) Moreover, more efficient encoding $\rightsquigarrow$ no ϵ-term

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Intersection Non-Emptiness

Theorem (Tally case, F & Krebs 2017)

There is no algorithm that, given k tally DFAs (or NFAs) $A_1, \ldots, A_k$, each with at most q states, decides if $\bigcap_{i=1}^{k} L(A_i) \neq \emptyset$ in time $\mathcal{O}^*(2^{o(\min(k, q^{1/2}))})$ unless ETH fails.

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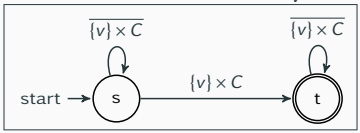
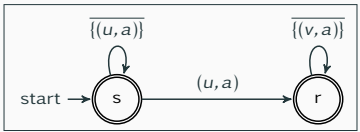
Reduction from 3-Coloring: Choose alphabet $\Sigma = V \times C$, $C = \{1, 2, 3\}$.

For all vertices v, build A_v:



For all edges uv, colors a, build $A_{uv,a}$:



Further ETH-based Automata Results: Synchronization

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Given a deterministic finite semi-automaton, i.e., for each $a \in \Sigma$, a mapping $f_a : Q \to Q$, a set $Q_{sync} \subseteq Q$, a Q_{sync}-*synchronizing word* $w \in \Sigma^*$ enjoys

$$\forall q, q' \in Q_{sync} : f_w(q) = f_w(q').$$

Q_{sync}-SW: Is there a Q_{sync}-*synchronizing word*? PSPACE-complete.
$Q_{sync} = Q$: Related to Černý's Conjecture.
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Theorem
There is an algorithm for solving Q_{sync}-SW on unbounded input alphabets that runs in time $\mathcal{O}^*(2^q)$ for q-state deterministic finite semi-automata. (Power automaton construction)
Conversely, assuming ETH, there is no $\mathcal{O}^*(2^{o(q)})$-time algorithm for this task, even on ***bounded*** *input alphabets.*

Further ETH-based Automata Results: Pumping

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Recall: The Pumping Lemma according to Kozen:
For any regular language $L \subseteq \Sigma^*$, there is a constant p_L such that: If $w \in L$ and $|w| \geq p_L$, then there is a decomposition $w = xyz$ with $x, z \in \Sigma^*$ and $y \in \Sigma^+$ such that $xy^*z \subseteq L$.

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Theorem (Holzer and Rauch, 2023)
Given an s-state DFA with input alphabet Σ s.t. $|\Sigma| \in \mathcal{O}(s)$, the smallest pumping constant cannot be approximated

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Complexity Results for Finite Automata Based on SETH, OVH, 3SUM

SETH Bounds

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There is no $\mathcal{O}^*((|\Sigma| - \varepsilon)^{\ell})$-time algorithm solving Q-SW (for any $\varepsilon > 0$) unless SETH fails, where ℓ upper-bounds the length of a synchronizing word.

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Similar results for Non-Universality, Inequivalence.

Theorem (Wehar 2016)
There is no $\mathcal{O}(n^{k-\varepsilon})$ algorithm solving DFA Intersection Non-Emptiness (for any $\varepsilon > 0$), where n is the sum of the number of states of all involved DFA, unless SETH fails.

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OVH Bounds

NFA Unambiguity (NFA-Unamb)

Input: NFA A

Problem: Has every input word at most one accepting run?

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Algorithm for NFA-Unamb:
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The proof involves strengthening Wehar's theorem. It makes use of the k-Orthogonal Vector Hypothesis which generalizes OVH to selecting k orthogonal vectors; k-OVH is also implied by SETH.

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3SUM Bounds

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The *dynamic acceptance problem* amounts to designing a data structure that can be initialized for a given timed automaton A, and afterwards, upon consuming consecutive elements of the data stream, efficiently maintains the information on whether the word read so far is accepted by A.

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Theorem
If the REAL 3SUM Hypothesis holds, then there is a two-clock timed automaton A (with additive constraints) such that there is no data structure that, when initialized on A, supports dynamic acceptance in time $\mathcal{O}(n^{1-\delta})$ for any $\delta > 0$, where n is the length of the consumed stream prefix.

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Parsing

Parsing Literature

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Parsing Like Valiant

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1. Re-define matrix multiplication: Let A, B be two $n \times n$ matrices whose entries are subsets of the NT set N. (Finally, $n = |w| + 1$.) Define $C = AB$ by

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These links transfer to generalized parsing, e.g., for Boolean grammars and for tree-adjoining grammars.

Automata Hypotheses for Fine-Grained Complexity

NFA Acceptance Hypothesis

NFA Acceptance (NFA-Acc)

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Support 1: NFA-Acc implies Online Matrix-Vector Multiplication Hypothesis (OMV) (Henzinger *et al.*)
Support 2: OVH implies NFA-Acc for sparse NFA, i.e., $|A| \in \mathcal{O}(|Q_A|)$
Support 3: NFA-Acc is equivalent to many other problems, e.g., in databases.

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Summary

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  - Take your favorite computationally hard formal language problem and prove that well-known algorithms are conditionally optimal. It sometimes helps to re-analyze known reductions.
  - Take your favorite "base problem" from fine-grained complexity and look for "corresponding" formal language problems. Recall that automata are also graphs ...
- There are also a lot of naturally open questions that can be generated in this way.
  There are also lots of examples provided in this talk.