Fine-grained Complexity of Ambiguity Testing of Automata

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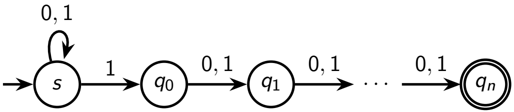
ETH Zürich

Joint work with Karolina Drabik, Fabian Frei, Filip Mazowiecki and Karol Węgrzycki
Warsaw University, CSAIL MIT, Max Planck Institute for Informatics

Finite Automata

Finite Automaton



$$\mathcal{A} = (Q, \delta, \Sigma, \{s\}, \{q_n\})$$
$$\mathcal{L}(\mathcal{A}) = (0+1)^*1(0+1)^n$$

- A word $w \in \Sigma^*$ is accepted by $\mathcal{A}$ if it labels an accepting run

DFA

✓ efficient for complementation, minimization, acceptance, containment, equivalence ...

NFA (this talk: no ε-transitions)

✗ containment and equivalence are PSPACE-complete

✓ exponentially more succint

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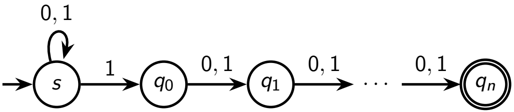

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- words label a unique accepting run

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✗ containment and equivalence are PSPACE-complete

✓ exponentially more succint

- words can label multiple accepting runs

Ambiguity of Finite Automata

Definition

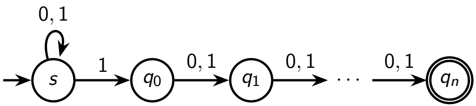
An NFA $\mathcal{A}$ is:

- **unambiguous** if every word has $\leqslant 1$ accepting run. $\rightarrow$ UFA

UFA

✓ UFA is more succint than DFA

✓ containment and equivalence are still polytime in UFA



Ambiguity of Finite Automata

Definition

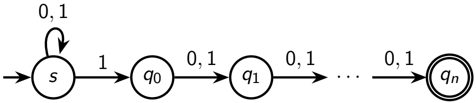
An NFA $\mathcal{A}$ is:

- **unambiguous** if every word has $\leqslant 1$ accepting run. $\rightarrow$ UFA
- **finitely ambiguous** if every word has $\leqslant$ constant accepting runs. $\rightarrow$ FFA
- **polynomially ambiguous** if every word has $\leqslant$ poly(|word|) accepting runs. $\rightarrow$ PFA

UFA

✓ UFA is more succint than DFA

✓ containment and equivalence are still polytime in UFA



$$DFA \subseteq UFA \subseteq FFA \subseteq PFA \subseteq NFA$$

Ambiguity Testing

Problem (Ambiguity Testing)

Given an NFA $\mathcal{A}$, is $\mathcal{A}$ unambiguous / finitely ambiguous / polynomially ambiguous?

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Deciding that $\mathcal{A}$ is	**Running time** [1991, 2001]
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finitely ambiguous (FFA)	n^3
polynomially ambiguous (PFA)	n^2

$n = |Q| + |\delta| =$ size of $\mathcal{A}$

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Problem (DISJOINTNESS)

Given two DFAs $\mathcal{D}_1$ and $\mathcal{D}_2$, decide whether $\mathcal{L}(\mathcal{D}_1 \cap \mathcal{D}_2) = \varnothing$.

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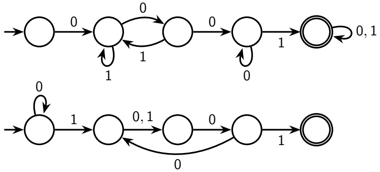
Disjointness → Unambiguity

- Given $\mathcal{D}_1$ and $\mathcal{D}_2$, let $\mathcal{A} \coloneqq \mathcal{D}_1 \cup \mathcal{D}_2$.
- $\mathcal{A}$ is unambiguous $\Leftrightarrow \mathcal{L}(\mathcal{D}_1 \cap \mathcal{D}_2) = \varnothing$



Improved lower bounds

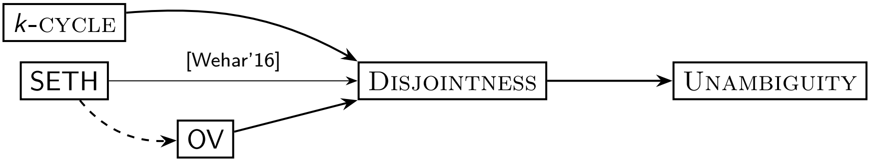
Orthogonal Vector

Given sets $A, B \subseteq \{0,1\}^d$ of size n decide if

$$\exists a \in A, \exists b \in B: \ a \perp b.$$

Hypothesis: $\forall \varepsilon > 0, \exists c > 0$ not in time $\mathcal{O}(n^{2-\varepsilon})$ for $d = c \log n$

Improved lower bounds



k-cycle

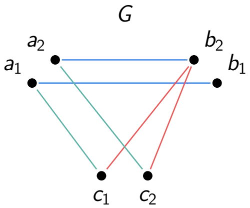
Given a k-partite graph $G = (A_1 \uplus \cdots \uplus A_k, E)$, decide if it contains a cycle of length k.
Hypothesis: $\forall \varepsilon > 0, \exists k \in \mathbb{N}$ not in time $\mathcal{O}(|E|^{2-\varepsilon})$

k-Cycle $\rightarrow$ Disjointness

Construction Given k-partite graph $G = (A_1 \uplus \cdots \uplus A_k, E)$, need to detect k-cycle.

Example
$k = 3$

$n = 2$



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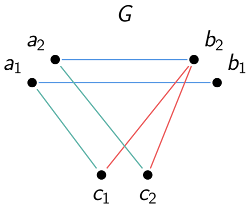

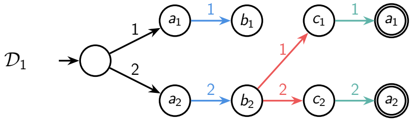

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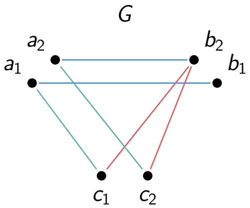

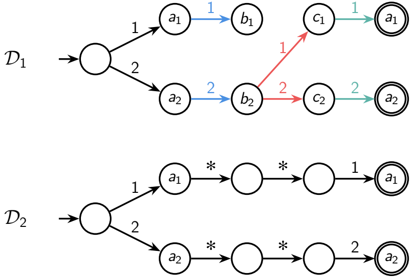

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→ Then any word $w \in \mathcal{L}(\mathcal{D}_1) \cap \mathcal{L}(\mathcal{D}_2)$ encodes a k-cycle in G.

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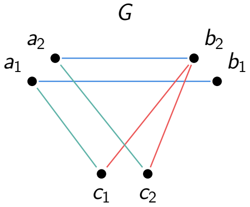

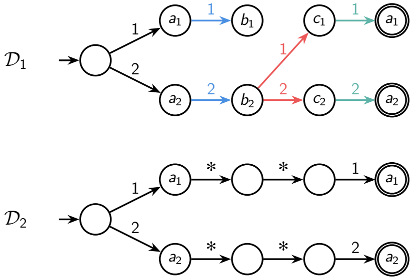

Size $\mathcal{O}(n + m)$ states and transitions

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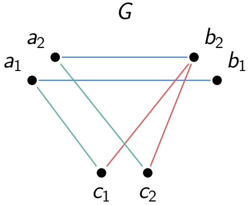
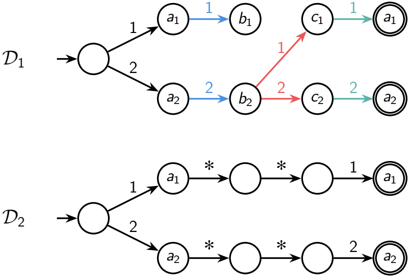
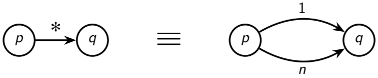




Issue



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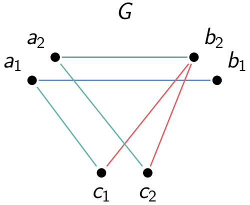
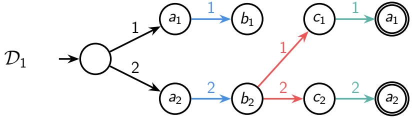
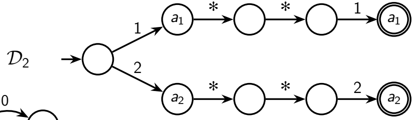
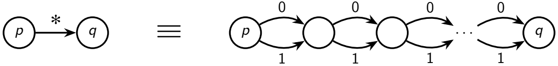






Fix



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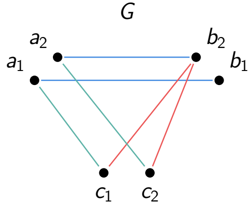

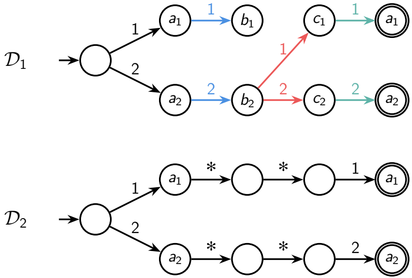

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